

THE FAILURE RATE IN RELIABILITY: APPROXIMATIONS AND BOUNDS¹

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ABSTRACT

We consider models typical to the area of reliability, and a failure rate function for processes describing the dynamics of these models. Various approximations of the failure rate function are proposed and their accuracies are investigated. The basic case studied in the paper is a regenerative model. Some interesting particular cases (Markov, semi-Markov, etc.) are considered. All proposed estimates are stated in a tractable analytic form.

This article is dedicated to the memory of Roland L. Dobrushin.

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1. Introduction

Like many other great scientists, R.L. Dobrushin had a knack for getting to the root of a problem due to his belief that only a simple and clear idea provides a foundation for studying any phenomena. His fantastic mathematical technique allowed him to convert such ideas into brilliant solutions of intricate problems.

This paper contains an example of how a general idea can be transformed into a solution leading to numerical results. We took a particular problem from the area of reliability that consists of the comparison of two failure rate intensities. Such a problem can be solved directly given additional assumptions imposed on the underlying system (say, that the system is Markovian). But such a solution is unsatisfactory, in general, because the additional assumptions lead to use tools which are not the best.

Our approach can be outlined in abstract terms as follows. First, we express the desired goal function in terms of auxiliary characteristics of the underlying process (which can be estimated

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separately) without imposing unnecessary restrictions. For example, the difference between failure rate intensities under comparison (a goal function) can be expressed in terms of passage times of various auxiliary processes (auxiliary characteristics). Second, we give estimates of auxiliary characteristics by any available method (analytical, numerical, or simulation). The choice of the method is dictated, in general, by the type of the underlying process. Note that the additional assumptions arise on the second stage only, while the expression of the goal function via auxiliary characteristics is obtained under general assumptions at the first stage. This enables us to enlarge the class of investigated systems.

The paper is concerned specifically with a so-called failure rate function that plays an important role in reliability and represents a conditional density to fail at time t provided that there was no failure until t . It seems clear to us and is reflected in the approach that before we can establish distributional properties of random processes, we must first study their structural features (such as regeneration or Markov property) and pathwise properties.

Throughout the paper, we deal with various random processes defined on $[0, \infty)$. It is convenient to denote by (η_s) a random process having state η_s at time $s \in [0, \infty)$. Sometimes (when this cannot lead to a confusion) we will use a shorter notation $\eta \equiv (\eta_s)$. The notation $(\eta_s)_{0 \leq s \leq t}$ refers to the restriction of (η_s) to the time segment $[0, t]$ (so, $(\eta_s) \equiv (\eta_s)_{0 \leq s < \infty}$).

Let a random process (η_s) describe the dynamics of a system and its state space E be partitioned into two subsets: $E = \mathcal{M} \cup \mathcal{P}$, $\mathcal{M} \cap \mathcal{P} = \emptyset$. We will view \mathcal{M} as a subset of “good” (or operating) states and \mathcal{P} as a subset of “bad” (or failed) states assuming that E is a complete separable metric space and all paths of (η_s) are *cadlag* (that is they are right-continuous and have limits from the left). Additional restrictions on process (η_s) will be imposed later on. Let

$$\sigma = \inf\{s: \eta_s \in \mathcal{P}\} \quad (1.1)$$

be the first break-down time (or lifetime) of the system. The following characteristic

$$\lambda(t) = \lim_{\Delta \rightarrow 0} \frac{\mathbf{P}(\sigma \leq t + \Delta \mid \sigma > t)}{\Delta} \quad (1.2)$$

is called the *failure rate function* for process (η_s) . Of course, it is nothing else but a *hazard rate function* for random lifetime σ . We assume that $\lambda(t)$ is defined correctly by relation (1.2). Unfortunately, function $\lambda(t)$ defined this way cannot be calculated easily, even for simple systems that are of interest in engineering. Even if we approximate lifetime σ by another random variable (r.v.) σ' which is close to σ , for example in a weak topology, this does not guarantee that the hazard rate function for σ' is close to $\lambda(t)$ in a natural sense.

Because of this, the following characteristic, called the *Vesely failure rate*, is often used instead of $\lambda(t)$:

$$\lambda_V(t) = \lim_{\Delta \rightarrow 0} \frac{\mathbf{P}(\eta_{t+\delta} \in \mathcal{P} \mid \eta_t \in \mathcal{M})}{\Delta} \quad (1.3)$$

(see Coccozza-Thivent and Roussignol [4] and [5]). These two functions $\lambda(t)$ and $\lambda_V(t)$ have different meaning. While $\lambda(t)$ characterizes the behavior of process (η_s) until its first entrance into the subset \mathcal{P} , function $\lambda_V(t)$ is concerned with the behavior of (η_s) even after its first entrance into \mathcal{P} . Nevertheless, in many applications $\lambda_V(t)$ serves as an approximation of $\lambda(t)$ and it is important to give estimates of the accuracy of this approximation. In this paper, we develop a general approach for obtaining such estimates, give corresponding examples, and propose another approximation $\lambda^0(t)$ for $\lambda(t)$ which depends on the behavior of (η_s) until its first entrance into \mathcal{P} .

In all considerations, we deal with limiting (as $t \rightarrow \infty$) characteristics, namely with $\lambda(\infty)$, $\lambda_V(\infty)$, and $\lambda^0(\infty)$, assuming that these limits exist. These characteristics are quite typical for reliability systems. Moreover, the usage of limiting characteristics instead of the prelimiting ones can be justified (see Coccozza-Thivent and Roussignol [4]).

The paper is organized as follows. In Section 2 we consider a decomposition which is a key tool in obtaining basic results. Examples of the decomposition are given. The core of the decomposition is the representation of the dynamics of (η_s) (until its first entrance to \mathfrak{P}) as the dynamics of a process (η_s^0) (obtained from (η_s) by tabooing transitions to the subset \mathfrak{P} of failed states) subjected to failures. These failures form a Cox process (a Poisson process with an intensity α_t modulated by (η_s^0) ; that is, $\alpha_t = \alpha((\eta_s^0)_{0 \leq s \leq t})$). Such a decomposition is used in Section 3 to obtain the failure rate function $\lambda(t)$ (see (1.2)) in terms of the intensity α_t . In turn, this enables us (in subsection 3.1) to write an equation determining $\lambda(\infty)$ in the case where (η_s^0) is a regenerative process with inter-regeneration times having an exponential moment. In subsection 3.2 we propose an upper bound λ^0 for $\lambda(\infty)$ which is an expectation of α_t with respect to the stationary distribution of (η_s^0) and we prove that $\lambda^0 = \lambda_V(\infty)$ in particular cases (the resulting inequality $\lambda(\infty) \leq \lambda_V(\infty)$ is very useful in practical studies). Subsection 3.3 deals with the Vesely failure rate (1.3) and its expression in terms of stationary characteristics of the process (η_s) .

The remainder of the paper refers to the case where intensity α_t depends on the current state of (η_s^0) : $\alpha_t = \alpha(\eta_t^0)$. In this case, $\lambda(\infty)$ can be represented as an expectation of $\alpha(\eta)$ with respect to a specific measure $\tilde{\pi}(\cdot)$. Section 4 contains a construction of a regenerative process $\tilde{\eta}$ for which $\tilde{\pi}$ is a stationary distribution. Using the mentioned constructions, we give accuracy estimates of λ^0 and λ_V in Section 5, under different restrictions imposed on process (η_s) .

Various examples illuminating the estimates proposed will appear in [6] as the second part of this work. These examples are motivated by their applications in reliability area. In particular, they show how the derived estimates can be used in engineering.

2. Decomposition of the Initial Process

2.1 Definitions

Throughout the paper, we will assume that all random objects are defined on a *common probability space* $(\Omega, \mathfrak{F}, \mathbf{P})$ with elementary outcomes $\omega \in \Omega$. But the argument ω will be skipped in most cases.

The following construction is crucial for us. Consider a random process (η_s^0) with state space \mathcal{M} and a random process (M_s) such that

- (1) for every t , the r.v. M_t is a function of $(\eta_s^0)_{0 \leq s \leq t}$;
- (2) for every t , the r.v. M_t is non-negative;
- (3) the functional M_t is a nondecreasing function of t .

In other words, if \mathfrak{F}_t^0 is the σ -field generated by random variables η_s^0 , $0 \leq s \leq t$, and $\mathfrak{F}^0 = (\mathfrak{F}_t^0)_{t \geq 0}$ is the natural filtration generated by the history of the process (η_s^0) , then (M_t) is an increasing process adapted to the filtration \mathfrak{F}^0 (see Jacod and Shiryaev [13] (Chapter 1)). Let us call (M_t) an *integral intensity function*. As we will see, the integral intensity function will be typically represented as the integral

$$M_t = \int_0^t \alpha_u du \tag{2.1}$$

with respect to the Lebesgue measure, and the *intensity function* $\alpha_u = \alpha_u((\eta_s^0)_{0 \leq s \leq u})$ is a non-negative and right-continuous process, adapted to the filtration \mathfrak{F}^0 . To avoid pathological cases, let us assume in this paper that representation (2.1) holds true. In fact, all results below can be reformulated in terms of M_t only, and therefore the requirement (2.1) can be relaxed.

Definition 2.1: Say that $((\eta_s, \sigma))$ can be decomposed into $((\eta_s^0), (\alpha_s))$ if

$$\mathbf{P}((\eta_s)_{0 \leq s \leq t} \in \cdot, \sigma > t) \equiv \mathbf{E}I((\eta_s^0)_{0 \leq s \leq t} \in \cdot, \sigma > t)$$

$$= \mathbf{E} \left(I((\eta_s^0)_{0 \leq s \leq t} \in \cdot) \exp \left(- \int_0^t \alpha_u du \right) \right) \tag{2.2}$$

for any $t \geq 0$.

To clarify equation (2.2), let us define an r.v. T jointly with (η_s^0) by the conditional distribution

$$\mathbf{P}(T > t \mid (\eta_s^0)) = \exp(-M_t) = \exp \left(- \int_0^t \alpha_u du \right). \tag{2.3}$$

It follows from the restrictions imposed on α_u that, for any $t \geq 0$,

$$\mathbf{P}(T > t \mid (\eta_s^0)) = \mathbf{P}(T > t \mid (\eta_s^0)_{0 \leq s \leq t}) \equiv \mathbf{P}(T > t \mid \mathcal{F}_t^0). \tag{2.4}$$

Thus α_t can be viewed as a random conditional failure rate of r.v. T at time t given $(\eta_s^0)_{0 \leq s \leq t}$. Equation (2.2) is equivalent to

$$\mathbf{P}((\eta_s)_{0 \leq s \leq t} \in \cdot, \sigma > t) = \mathbf{P}((\eta_s^0)_{0 \leq s \leq t} \in \cdot, T > t). \tag{2.5}$$

This last equation will play a role similar to the coupling used in Cocozza-Thivent and Roussignol [4].

The following lemma is an obvious but useful consequence of Definition 2.1.

Lemma 2.2: *Suppose that $((\eta_t), \sigma)$ can be decomposed into $((\eta_t^0), (\alpha_t))$ and that*

$$0 \leq \alpha_t \leq \epsilon, \quad \forall t \geq 0.$$

Then,

$$\mathbf{P}(\sigma > t) \geq e^{-\epsilon t}, \quad \forall t \geq 0.$$

This lemma means that σ is dominated stochastically by an r.v. having an exponential law with parameter ϵ if the intensity function is bounded by ϵ . The notion of stochastic domination can be found in Shaked and Shanthikumar [21] (Section 1.A) or Lindvall [18] (Chapter IV.1).

2.2 Examples

In this subsection we treat a variety of models partly motivated by their applications to reliability. They illustrate Definition 2.1 and contain corresponding decompositions which will be used in the sequel.

Example 2.3: Markov process on a discrete state space. Consider a time-homogeneous Markov process (η_s) on a discrete state space E with transition rates

$$A(i, j) = \lim_{\Delta \rightarrow 0} \frac{\mathbf{P}(\eta_{t+\Delta} = j \mid \eta_t = i)}{\Delta}, \quad i \neq j, \quad i, j \in E. \tag{2.6}$$

Let (η_s^0) be a Markov process with the same initial distribution as (η_s) and transition rates

$$A^0(i, j) = A(i, j), \quad i \neq j, \quad i, j \in \mathcal{M}. \tag{2.7}$$

Define intensity α_t as

$$\alpha_t = A(\eta_t^0, \mathcal{P}), \tag{2.8}$$

where

$$A(i, \mathcal{P}) = \sum_{j \in \mathcal{P}} A(i, j).$$

Then decomposition (2.2) holds true; the proof will be given later for a more general case (see Examples 2.4 and 2.9). This decomposition is widely used in the theory of Markov processes, and in that case Lemma 2.2 is quite intuitive.

Example 2.4: Semi-Markov process on a discrete state space. Let $(X, T) = (X_n, T_n)_{n \geq 0}$ be a Markov renewal process on a discrete state space E having a semi-Markov kernel with density q , that is

$$\mathbf{P}(X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i) = \int_0^t q(i, j, x) dx.$$

Denote by (η_s) the semi-Markov process associated with (X, T) and defined by the equality

$$\eta_t = X_n \text{ if } T_n \leq t < T_{n+1}.$$

The conditional density of T_1 , given $X_0 = i$, has the form

$$f(i, t) = \sum_{j \in E} q(i, j, t),$$

and its hazard rate function is equal to

$$h(i, t) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{P}(T_1 \leq t + \Delta \mid X_0 = i, T_1 > t) = \frac{f(i, t)}{\int_t^\infty f(i, u) du}.$$

Let

$$p(i, j, t) = \frac{q(i, j, t)}{f(i, t)} = \mathbf{P}(X_1 = j \mid X_0 = i, T_1 = t).$$

Then the density of the semi-Markov kernel can be represented as

$$q(i, j, t) = p(i, j, t)f(i, t) = p(i, j, t)h(i, t) \exp\left(-\int_0^t h(i, s) ds\right).$$

So the transition rate function,

$$A(i, j, t) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{P}(X_1 = j, T_1 \leq t + \Delta \mid X_0 = i, T_1 > t),$$

can be expressed as

$$A(i, j, t) = p(i, j, t)h(i, t). \tag{2.9}$$

Since $\sum_{j \in E} p(i, j, t) = 1$,

$$h(i, t) = \sum_{j \in E} A(i, j, t)$$

and therefore,

$$q(i, j, t) = A(i, j, t) \exp\left(-\int_0^t \sum_{k \in E} A(i, k, s) ds\right). \tag{2.10}$$

It follows from relation (2.10) that a semi-Markov process can be defined either by its kernel or by transition rate function.

The decomposition of a semi-Markov process is given in the following proposition.

Proposition 2.5: Let (η_s) be a semi-Markov process with a finite state space E and transition rate function $A(i, j, x)$ and suppose $\eta_0 \in \mathcal{M}$. For $\mathfrak{P} \subset E$ define

$$A(i, \mathfrak{P}, t) = \sum_{j \in \mathfrak{P}} A(i, j, t)$$

and $\sigma = \inf\{t: \eta_t \in \mathfrak{P}\}$. Let (η_s^0) be a semi-Markov process with state space \mathcal{M} , the same initial distribution as (η_s) and transition rates

$$A_0(i, j, t) = A(i, j, t), \quad i \neq j, \quad i, j \in \mathcal{M}.$$

Define the intensity function α_t by the equality

$$\alpha_t = A(\eta_t^0, \mathfrak{P}, y_t^0) \tag{2.11}$$

where y_t^0 is the time spent by process (η_s^0) in its current state.

Then $((\eta_t), \sigma)$ can be decomposed into $((\eta_t^0), (\alpha_t))$.

Sketch of the Proof: It is sufficient to show that, for any $0 < t_1 < \dots < t_k < t_{k+1} = t$ and i_1, \dots, i_k in \mathcal{A} ,

$$\mathbf{P}(\eta_{t_1} = i_1, \dots, \eta_{t_k} = i_k, \sigma > t) = \mathbf{E} \left(I(\eta_{t_1}^0 = i_1, \dots, \eta_{t_k}^0 = i_k) \exp \left(- \int_0^t \alpha_u du \right) \right). \quad (2.12)$$

Introducing the Markov renewal process (X, T) and conditioning by $\eta_0 = X_0 = i_0$, we obtain

$$\begin{aligned} & \mathbf{P}(\eta_{t_1} = i_1, \dots, \eta_{t_k} = i_k, \sigma > t \mid \eta_0 = i_0) \\ &= \sum_{0 \leq n_1 \leq \dots \leq n_k \leq n} \mathbf{P} \left(\bigcap_{j=1}^k \{X_{n_j} = i_j, T_{n_j} \leq t_j < T_{n_j+1}\}, T_n \leq t < T_{n+1}, \sigma > t \mid X_0 = i_0 \right) = \\ & \sum_N \mathbf{P} \left(\bigcap_{r=0}^k \left(\{X_{n_{r+1}} = i_{r+1}, T_{n_{r+1}} \leq t_{r+1} < T_{n_{r+1}+1}\} \bigcap_{l=1}^{n_{r+1}-n_r-1} \{X_{n_r+l} \in \mathcal{A}\} \right) \mid X_0 = i_0 \right) \\ &= \sum_N \sum_{r=0}^k \sum_{l=1}^{n_{r+1}-n_r-1} \sum_{x_{r+1}, l \in \mathcal{A}} \int \dots \int \left(\prod_{r=0}^k q(i_r, x_{r+1}, l, u_{n_r+1}) \right. \\ & \quad \left. \times \prod_{l=1}^{n_{r+1}-n_r-1} q(x_{r+1}, l, x_{r+1}, l+1, u_{n_r+l+1}) \right) \int_{t-u_n}^{\infty} f(x_{k+1}, n-n_k-1, s) ds \\ & \quad \times I \left(\bigcap_{j=1}^k \{u_1 + \dots + u_{n_j} \leq t_j < u_1 + \dots + u_{n_j+1}\}, u_1 + \dots + u_n \leq t \right) du_1 \dots du_n, \quad (2.13) \end{aligned}$$

where the outer sum \sum_N is taken over the set $N = \{(n_0, \dots, n_{k+1}; 0 = n_0 \leq n_1 \leq \dots \leq n_k \leq n_{k+1} = n)\}$. Let q_0 be the Markov renewal kernel density of process (η_s^0) . Equation (2.10) and a similar equation written for process (η_s^0) show that

$$q(i, j, u) = q_0(i, j, u) \exp \left(- \int_0^u A(i, \mathfrak{P}, s) ds \right). \quad (2.14)$$

If h_0 is the hazard rate of the first jump time for process (η_s^0) , then

$$\int_u^{+\infty} f(i, s) ds = \exp \left(- \int_0^u h(i, s) ds \right) = \exp \left(- \int_0^u (h_0(i, s) + A(i, \mathfrak{P}, s)) ds \right). \quad (2.15)$$

Substituting (2.14) and (2.15) into (2.13) we arrive at the desired formula (2.12). \square

Example 2.6: Redundant system. Let a system consist of N elements. Each element can be in one of the two states 0 or 1. State 0 corresponds to the failure of an element while state 1 means that the element is operating. Assume that all elements are independent and let $\eta_t = (\eta_t(1), \dots, \eta_t(N))$ be the state vector of the system. Assume additionally that each marginal process $\eta_t(i)$, $1 \leq i \leq N$, is an *alternating renewal process*. Then successive sojourn times of $\eta_t(i)$ at states 0 and 1 are independent r.v.'s. Denote by $\lambda^{(i)}(t)$ and $\mu^{(i)}(t)$ failure and repair rates of element i ($1 \leq i \leq N$), respectively. In these terms, the density function of a sojourn time at state 1 (at state 0, resp.) for element i can be represented in the form

$$f_i(t) = \lambda^{(i)}(t) \exp \left(- \int_0^t \lambda^{(i)}(s) ds \right)$$

$$\text{resp. } g_i(t) = \mu^{(i)}(t) \exp\left(-\int_0^t \mu^{(i)}(s) ds\right).$$

Let

$$y_t = (y_t(1), \dots, y_t(N)) \tag{2.16}$$

be the vector of elapsed time of elements in their current states. For instance, $y_t(i)$ is the time elapsed from the last (before t) jump of $(\eta_s(i))$. In fact, y_t can be viewed as a functional of $(\eta_s)_{0 \leq s \leq t}$. Process (η_s) can be regarded as a piecewise-constant process with jumps defined by the rates

$$\lambda^{(i)}(y_t(i)) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{P}(\eta_{t+\Delta}(j) = \eta_t(j), \forall j \neq i; \eta_{t+\Delta}(i) = 0 \mid (\eta_s)_{0 \leq s \leq t}, \eta_t(i) = 1)$$

and

$$\mu^{(i)}(y_t(i)) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{P}(\eta_{t+\Delta}(j) = \eta_t(j), \forall j \neq i; \eta_{t+\Delta}(i) = 1 \mid (\eta_s)_{0 \leq s \leq t}, \eta_t(i) = 0).$$

Now, let us assume that process (η_s) can reach set \mathfrak{P} from \mathcal{M} only at times when one of the element fails. Let $\eta = (\eta(1), \dots, \eta(N))$ be a vector of binary components denoting a state of the system and let

$$R_1(\eta) = \{k: \eta(k) = 1, 1 \leq k \leq N\}$$

be the set of operating elements. For $\eta \in \mathcal{M}$, define another subset of elements (that may be empty)

$$R(\eta) \subset R_1(\eta), \tag{2.17}$$

such that

$$k \in R(\eta) \Leftrightarrow \eta(k) = 1, (\eta(1), \dots, \eta(k-1), 0, \eta(k+1), \dots, \eta(N)) \in \mathfrak{P}.$$

In other words, $R(\eta)$ is the set of ‘‘critical components’’ with respect to state η . Define a piecewise-constant process (η_s^0) with state space \mathcal{M} by the rates

$$\lambda^{(i)}(y_t^0(i)) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{P}(\eta_{t+\Delta}^0(j) = \eta_t^0(j), \forall j \neq i; \eta_{t+\Delta}^0(i) = 0 \mid (\eta_s^0)_{0 \leq s \leq t}, \eta_t^0(i) = 1)$$

and

$$\mu^{(i)}(y_t^0(i)) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{P}(\eta_{t+\Delta}^0(j) = \eta_t^0(j), \forall j \neq i; \eta_{t+\Delta}^0(i) = 1 \mid (\eta_s^0)_{0 \leq s \leq t}, \eta_t^0(i) = 0),$$

where vector y_t^0 of elapsed times for (η_s^0) is defined quite similarly as y_t . Let

$$\alpha_s = \sum_{i \in R(\eta_s^0)} \lambda^{(i)}(y_s^0(i)). \tag{2.18}$$

Then condition (2.2) is met.

The proof of this result can be given by similar arguments as in the previous example. We will exhibit another approach in the following subsection and give an alternative proof for this result (see Example 2.11).

Observe that, in reliability theory, one often considers decomposed processes, where (η_t^0) describes the system dynamics *without failing factors*. The failure occurs within $[t, t + \Delta)$ with probability $\alpha_t \Delta$ and the intensity α_t to fail depends either on state η_t^0 or even on prehistory $(\eta_s^0)_{0 \leq s \leq t}$.

2.3 A general method

We now present a general approach that can be applied for time-homogeneous Markov processes (η_s) . Some aspects that are intuitively reasonable are taken at face value, in order that the basic idea may be exposed without the burden of too much mathematical detail. The assumption that (η_s) is a Markov process is not restrictive since a non-Markov process can be embedded into a Markov process by equipping it with supplementary coordinates (see Example 2.11 and

Kalashnikov [15] (Chapter 10)). Process (η_s) can be defined in terms of its transition probability

$$Q(\eta; t, \cdot) = \mathbf{P}(\eta_{s+t} \in \cdot \mid \eta_s = \eta). \quad (2.19)$$

However, this characteristic is not convenient for practical purposes since it cannot be found in a closed form for many processes of interest. In practice, one prefers to define Markov processes with the help of so-called *infinitesimal characteristics*. One of the most useful infinitesimal characteristics is the *generator* \mathbf{A} . A naive definition of the generator can be given as follows (see Dynkin [9] (Chapters 1 and 2), Feller [11] (Chapter X), and Ethier and Kurtz [10] (Chapter 4) for rigorous definitions). Let us view the transition probability (2.19) as an operator that maps the set of measurable bounded functions $f: E \rightarrow \mathbf{R}^1$ into itself by the formula

$$g_t(\eta) = \int_E f(\xi) Q(\eta; t, d\xi) = \mathbf{E}(f(\eta_t) \mid \eta_0 = \eta), \quad t \geq 0,$$

where $g_0(\eta) = f(\eta)$. The following characteristic is called the *infinitesimal operator* or *generator* of (η_s) :

$$g(\eta) \equiv \mathbf{A}f(\eta) = \lim_{t \rightarrow 0} \frac{g_t(\eta) - g_0(\eta)}{t} \quad (2.20)$$

provided the limit exists in the sup-normed topology $\|f\| = \sup_{\eta} |f(\eta)|$; $\mathbf{A}f(\eta)$ is a derivative of the mean value of $f(\eta_t)$ at $t = 0$, along the path of process (η_s) given $\eta_0 = \eta$; it is not defined on the set of all measurable bounded functions (in general) but only on a subset $D_{\mathbf{A}}$ called the *generator domain*. This subset depends on the type of process under consideration. Actually, there are several definitions of the generator owing to the variety of meanings of the limit in (2.20) (see Dynkin [9]) and due to the expansion of the domain of \mathbf{A} to unbounded functions (see Davis [7] and Kalashnikov [14]).

We restrict ourselves with only Feller processes (see Dynkin [9] (Chapter 2) and Meyn and Tweedie [19]). This class of processes can be characterized by the property that transition probabilities (2.19) map the set of bounded continuous functions into itself. It follows, in particular, that Feller processes have transition functions that are continuous with respect to the first argument (initial state).

Operator \mathbf{A} is linear and there is one-to-one correspondence between \mathbf{A} and transition probability Q under wide conditions (see Dynkin [9] (Chapter 1)). Here, we will assume that such a correspondence takes place. Formula (2.20) is equivalent to the equation

$$\mathbf{E}(f(\eta_h) \mid \eta_0 = \eta) = f(\eta) + h\mathbf{A}f(\eta) + o(h). \quad (2.21)$$

The following theorem gives a general way of decomposing a Markov process.

Theorem 2.7: *Let (η_s) be a Markov process with state space E and generator \mathbf{A} , and let \mathcal{M} be a subset of E , $\mathfrak{P} = \mathcal{M}^c$ and $\sigma = \inf\{t: \eta_t \in \mathfrak{P}\}$. Suppose that \mathbf{A} can be decomposed into the sum*

$$\mathbf{A} = \mathbf{A}^0 + \mathbf{A}^{\mathfrak{P}}, \quad (2.22)$$

such that $\mathbf{A}^0 f(\eta)$ does not depend on values $f(\xi), \xi \in \mathfrak{P}$, if $\eta \in \mathcal{M}$, and component $\mathbf{A}^{\mathfrak{P}}$ has the form

$$\mathbf{A}^{\mathfrak{P}} f(\eta) = \int_{\mathfrak{P}} f(\xi) A(\eta; d\xi) - f(\eta) A(\eta; \mathfrak{P}), \quad \eta \in E, \quad (2.23)$$

where $A(\eta; \cdot)$ is a finite positive measure with support \mathfrak{P} for each $\eta \in \mathcal{M}$, continuous with respect to $\eta \in \mathcal{M}$ and $\sup_{\eta \in \mathcal{M}} A(\eta, \mathfrak{P}) < \infty$. Then $((\eta_s), \sigma)$ can be decomposed into $((\eta_s^0), (\alpha_s))$, where (η_s^0) is a Markov process with generator \mathbf{A}^0 (restricted to \mathcal{M}) and

$$\alpha_s = A(\eta_s^0; \mathfrak{P}). \quad (2.24)$$

Proof: Formula (2.24) can be regarded as a consequence of more general results contained in

Dynkin [9] (Theorem 9.7 and Sections 10.1 to 10.3) if one treats (η_s) as a Markov process that exits a time σ . Our purpose here is to present an intuitive reasoning of the result rather than its rigorous version given in the cited book.

Let us estimate the probability

$$p_t(B) \equiv \mathbf{P}((\eta_s)_{0 \leq s \leq t} \in B; \sigma > t) \tag{2.25}$$

for fixed time $t > 0$ and subset B . Process (η_s) has right-continuous paths. Therefore, it is sufficient to find probability (2.25) for a cylindric set B that is to take B of the form

$$\{\omega: (\eta_s)_{0 \leq s \leq t}(\omega) \in B\} = \{\omega: \eta_{t_j}(\omega) \in B_j \subset \mathcal{M}, 1 \leq j \leq n, 0 \leq t_1 < t_2 < \dots < t_n \leq t\}, \tag{2.26}$$

where $n \geq 1$ and $0 \leq t_1 < t_2 < \dots < t_n \leq t$ are arbitrary.

Let us define a killed Markov process $(\tilde{\eta}_s)$ (see Dynkin [9], (Chapter 10)):

$$\tilde{\eta}_s = I(\sigma > s)\eta_s + I(\sigma \leq s)\phi. \tag{2.27}$$

This means that the state $\tilde{\eta}_s$ is not defined after stopping time σ . Denote by $\tilde{\mathbf{A}}$ the generator of $(\tilde{\eta}_s)$. In these terms,

$$p_t(B) \equiv \mathbf{P}(\tilde{\eta}_{t_j} \in B_j, 1 \leq j \leq n, \tilde{\eta}_t \in \mathcal{M}). \tag{2.28}$$

Let us take a function f from the domain of operator \mathbf{A}^0 that is equal to zero in \mathcal{P} . Then, by (2.22) and (2.23),

$$\mathbf{A}f(\eta) = \tilde{\mathbf{A}}f(\eta) = \mathbf{A}^0f(\eta) - A(\eta, \mathcal{P})f(\eta), \quad \eta \in \mathcal{M}. \tag{2.29}$$

This means (see (2.21)) that, for any $\eta \in \mathcal{M}$,

$$\begin{aligned} \mathbf{E}(f(\tilde{\eta}_h) | \tilde{\eta}_0 = \eta) &= f(\eta) + h\mathbf{A}^0f(\eta) - hA(\eta; \mathcal{P})f(\eta) + o(h) \\ &= (1 - A(\eta; \mathcal{P})h)(f(\eta) + h\mathbf{A}^0f(\eta)) + o(h) \\ &= e^{-A(\eta; \mathcal{P})h}\mathbf{E}(f(\eta_h^0) | \eta_0^0 = \eta) + o(h) \\ &= \mathbf{E}\left(f(\eta_h^0) \exp\left(-\int_0^h A(\eta_s^0; \mathcal{P})ds\right) | \eta_0^0 = \eta\right) + o(h). \end{aligned} \tag{2.30}$$

As there is one-to-one correspondence between transition functions and their generators, we have, for any $C \subset \mathcal{M}$,

$$\mathbf{P}(\tilde{\eta}_h \in C | \tilde{\eta}_0 = \eta) = \mathbf{E}\left(I(\eta_h^0 \in C) \exp\left(-\int_0^h A(\eta_u^0; \mathcal{P})du\right) | \eta_0^0 = \eta\right) + o(h). \tag{2.31}$$

Using the Markov property of processes $(\tilde{\eta}_s)$ and (η_s^0) and equality (2.31), and letting $h \rightarrow 0$, we have for any $s = mh, m \geq 1$,

$$\begin{aligned} \mathbf{P}(\tilde{\eta}_s \in C) &= \lim_{m \rightarrow \infty} \mathbf{E}\left(\prod_{j=0}^{m-2} \mathbf{P}(\tilde{\eta}_{(j+1)h} \in \mathcal{M} | \tilde{\eta}_j) \mathbf{P}(\tilde{\eta}_{mh} \in C | \tilde{\eta}_{s-h})\right) \\ &= \lim_{m \rightarrow \infty} \mathbf{E}\left(\mathbf{P}(\eta_s^0 \in C | \eta_{s-h}^0) \mathbf{E}\prod_{j=0}^{m-2} \exp\left(-\int_{jh}^{(j+1)h} A(\eta_u^0; \mathcal{P})du\right)\right) \\ &= \mathbf{E}\left(I(\eta_s^0 \in C) \exp\left(-\int_0^s A(\eta_u^0; \mathcal{P})du\right)\right). \end{aligned}$$

This and the Markov property imply that the probability (2.28) has the form

$$p_t(B) = \mathbf{E} \left(I(\eta_{t,j}^0 \in B_j, 1 \leq j \leq n) \exp \left(- \int_0^t A(\eta_u^0; \mathfrak{P}) du \right) \right).$$

This yields the result. □

Remark 2.8: It is useful to know that $\mathbf{1}_{\eta \in \mathcal{M}} \in D_{\mathbf{A}}$ and therefore $\mathbf{1}_{\eta \in \mathfrak{P}} = 1 - \mathbf{1}_{\eta \in \mathcal{M}} \in D_{\mathbf{A}}$. To prove this, note that $f(\eta) \equiv 1, \eta \in \mathcal{M}$, belongs to $D_{\mathbf{A}^0}$. This function f definitely belongs to the domain $D_{\bar{\mathbf{A}}}$ (where $\bar{\mathbf{A}}$ is defined in (2.29)) by (2.20), (2.21), and (2.22). In fact, $\tilde{\mathbf{A}}$ is the generator of the killed Markov process that exits at time σ and $D_{\bar{\mathbf{A}}} = D_{\mathbf{A}^0}$ by Theorem 9.3 from Dynkin [9]. Let us extend f as follows:

$$f_{\mathfrak{P}}(\eta) = \begin{cases} 1, & \text{if } \eta \in \mathcal{M}, \\ 0, & \text{if } \eta \in \mathfrak{P}. \end{cases}$$

Such an extended function belongs to $D_{\mathbf{A}}$ just because of the definition of generators $\tilde{\mathbf{A}}, \mathbf{A}$, and equality (2.29).

Process (η_s^0) may contain both jumpwise and continuous components (see Examples 2.10 and 2.11). But, due to form (2.23) of operator $\mathbf{A}^{\mathfrak{P}}$, process (η_s) may hit set \mathfrak{P} only by jumps, and the intensity of the jump at point $x \in E$ is equal to $A(x; \mathfrak{P})$. If such a jump leads to subset \mathfrak{P} , then the state of process (η_s) just after the jump is random with the probability distribution $A(x; \cdot)/A(x; \mathfrak{P})$, while process (η_s^0) stays in \mathcal{M} .

Example 2.9: Markov process on a discrete state space. Consider process (η_s) from Example 2.3. By definition (2.6), its generator has the form

$$\mathbf{A}f(i) = \sum_{j \in E} A(i, j)(f(j) - f(i)), \quad i \in E.$$

Let us decompose the operator \mathbf{A} in the following way:

$$\mathbf{A} = \mathbf{A}^0 + \mathbf{A}^{\mathfrak{P}},$$

where

$$\mathbf{A}^0 f(i) = \sum_{j \in \mathcal{M}} A(i, j)(f(j) - f(i)), \quad i \in E,$$

and

$$\mathbf{A}^{\mathfrak{P}} f(i) = \sum_{j \in \mathfrak{P}} A(i, j)(f(j) - f(i)), \quad i \in E.$$

Then, all conditions of Theorem 2.7 are fulfilled and formulas (2.8) and (2.24) coincide.

Example 2.10: Semi-Markov process on a discrete state space. Consider a semi-Markov process (η_s) from Example 2.4 and denote by y_s the elapsed time of the process in its current state at time s . Then process $\eta'_s = (\eta_s, y_s), s \geq 0$, is Markov and its generator has the form

$$\mathbf{A}f(i, y) = \frac{\partial f(i, y)}{\partial y} + \sum_{j \in E} A(i, j, y)(f(j, 0) - f(i, y)).$$

Let $\mathcal{M}' = \mathcal{M} \times [0, \infty)$ and $\mathfrak{P}' = \mathfrak{P} \times [0, \infty)$. This means that $(i, y) \in \mathcal{M}'$ if and only if $i \in \mathcal{M}$ and $(i, y) \in \mathfrak{P}'$ if and only if $i \in \mathfrak{P}$. The term $\partial f(i, y)/\partial y$ in the above formula is responsible for deterministic behavior of y_t between successive jumps of (η_s) , and, for $(i, y) \in \mathcal{M}'$, it *does not depend* on values of $f(j, z), (j, z) \in \mathfrak{P}'$. We arrive at the decomposition

$$\mathbf{A} = \mathbf{A}^0 + \mathbf{A}^{\mathfrak{P}}$$

with

$$\mathbf{A}^0 f(i, y) = \frac{\partial f(i, y)}{\partial y} + \sum_{j \in \mathcal{M}} A(i, j, y)(f(j, 0) - f(i, y)),$$

$$\mathbf{A}^{\mathfrak{P}} f(i, y) = \sum_{j \in \mathfrak{P}} A(i, j, y)(f(j, 0) - f(i, y)).$$

The expression for $A(\eta; \cdot)$ given in Theorem 2.7 (see equation (2.23)), for $\eta = (i, y)$, is equal to $A(i, \cdot, y)$ and formula (2.11) agrees with (2.24).

Example 2.11: Redundant system. Consider process (η_s) from Example 2.6. If $\eta'_s = (\eta_s, y_s)$, where y_s is defined by (2.16), then (η'_s) is a Markov process. It can be verified that

$$\mathbf{A}f(\eta, y) = \sum_{i=1}^N \frac{\partial f(\eta, y)}{\partial y_i} + \sum_{i=1}^N a^{(i)}(\eta, y)(f(\eta^{(i)}, y^{(i)}) - f(\eta, y)),$$

where $\eta = (\eta(1), \dots, \eta(N))$, $y = (y(1), \dots, y(N))$,

$$a^{(i)}(\eta, y) = \begin{cases} \lambda^{(i)}(y(i)), & \text{if } \eta(i) = 1, \\ \mu^{(i)}(y(i)), & \text{if } \eta(i) = 0, \end{cases}$$

$$\eta^{(i)}(j) = \eta(j), j \neq i; \quad \eta^{(i)}(i) = 1 - \eta(i);$$

$$y^{(i)}(j) = y(j), j \neq i; \quad y^{(i)}(i) = 0.$$

Let $\mathcal{M}' = \mathcal{M} \times [0, \infty)^N$ and $\mathfrak{P}' = \mathfrak{P} \times [0, \infty)^N$. Decompose operator \mathbf{A} in the same way as in Example 2.10

$$\mathbf{A} = \mathbf{A}^0 + \mathbf{A}^{\mathfrak{P}'},$$

where

$$\mathbf{A}^0 f(\eta, y) = \sum_{i=1}^N \frac{\partial f(\eta, y)}{\partial y_i} + \sum_{i \notin R(\eta)} a^{(i)}(\eta, y)(f(\eta^{(i)}, y^{(i)}) - f(\eta, y))$$

(the set $R(\eta)$ is defined in (2.17)) and

$$\mathbf{A}^{\mathfrak{P}'} f(\eta, y) = \sum_{i \in R(\eta)} a^{(i)}(\eta, y)(f(\eta^{(i)}, y^{(i)}) - f(\eta, y)).$$

In this case,

$$A((\eta, y); \mathfrak{P}') = \sum_{i \in R(\eta)} a^{(i)}(\eta, y). \tag{2.32}$$

Using Theorem 2.7, we arrive at formula (2.18).

3. The Asymptotic Failure Rate and its Approximations

3.1 The asymptotic failure rate

The failure rate $\lambda(t)$ and its limiting value $\lambda(\infty)$ provide useful characteristics for doing a number of calculations in reliability. In this section we obtain explicit expressions for them and their approximations. To state our results, let us assume that the intensity function $\alpha_t = \alpha_t((\eta_s^0)_{0 \leq s \leq t})$ satisfies the following conditions:

Conditions C(α)

1. $((\eta_s), \sigma)$ can be decomposed into $((\eta_s^0), (\alpha_s))$.
2. For each $t \geq 0$, function $\alpha_s \equiv \alpha_s(\omega)$ is right-continuous at $s = t$ for almost all $\omega \in \Omega$.
3. There exists an absolute constant $\bar{\alpha}$ such that

$$\sup_s \alpha_s = \bar{\alpha} < +\infty \tag{3.1}$$

The first condition is crucial to our approach. The second condition is quite natural because of the assumption that paths of (η_s^0) are cadlag. The third assumption is restrictive, of course. For example, it may fail for semi-Markov models when holding time have IFR distributions with unbounded intensities. Moreover, some final formulas in this work hold true in the case when conditions $C(\alpha)$ are violated. This means that the third assumption can be relaxed. It seems that the most straightforward and general way to do this is to find comparison estimates which yield that the desired characteristics of the process with unbounded function α_t can be obtained as the limit of corresponding characteristics for the process with intensity $\min(\alpha_t, \bar{\alpha})$ when $\bar{\alpha} \rightarrow \infty$. We will not touch this problem here but refer the reader to works of Kalashnikov [16] and [17] where such problems have been solved for specific characteristics of regenerative processes. Perhaps, comparison estimates wanted for the purposes of this paper may lead to different setups in comparison theorem.

We now express the failure rate function $\lambda(t)$ introduced in (1.2) in terms of the “decomposition components” of $((\eta_s, \sigma)$.

Lemma 3.1: *Let conditions $C(\alpha)$ hold. Then*

$$\lambda(t) = \frac{\mathbf{E} \left(\alpha_t \exp \left(- \int_0^t \alpha_u du \right) \right)}{\mathbf{E} \exp \left(- \int_0^t \alpha_u du \right)}.$$

Proof: We have, by (1.2) and (2.2),

$$\begin{aligned} \lambda(t) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \frac{\mathbf{P}(\sigma > t) - \mathbf{P}(\sigma > t + \Delta)}{\mathbf{P}(\sigma > t)} \\ &= \frac{1}{\mathbf{E} \exp \left(- \int_0^t \alpha_u du \right)} \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{E} \left(\exp \left(- \int_0^t \alpha_u du \right) - \exp \left(- \int_0^{t+\Delta} \alpha_u du \right) \right). \end{aligned}$$

By condition $C(\alpha)$.2, for every t there exists a right derivative $\frac{d}{dt} \int_0^t \alpha_u(\omega) du = \alpha_t(\omega)$ (a.s.). As

$$\frac{1}{\Delta} \left(\exp \left(- \int_0^t \alpha_u du \right) - \exp \left(- \int_0^{t+\Delta} \alpha_u du \right) \right) \leq \exp \left(- \int_0^t \alpha_u du \right) \sup_{t \leq s \leq t+\Delta} \alpha_s$$

for any $0 < \Delta \leq 1$, the Lebesgue dominated convergence theorem yields

$$\lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{E} \left(\exp \left(- \int_0^t \alpha_u du \right) - \exp \left(- \int_0^{t+\Delta} \alpha_u du \right) \right) = \mathbf{E} \left(\alpha_t \exp \left(- \int_0^t \alpha_u du \right) \right).$$

This completes the proof. □

Let us find sufficient conditions ensuring the existence of the limit

$$\lambda(\infty) = \lim_{t \rightarrow \infty} \lambda(t).$$

For this, we use a “regeneration technique” (see Kalashnikov [16]). Let us recall that a probability distribution ν is *spread out* if there exists an n such that ν^{*n} has a non-zero absolutely continuous component (see Asmussen [1] (Chapter VI)). A random variable is said to be *spread out* if its probability distribution is spread out.

Conditions C0

1. Conditions $C(\alpha)$ hold true.
2. Process (η_s^0) is regenerative (in the sense of Smith) with the sequence of regeneration times $0 = S_0, S_1, S_2, \dots$. Let $S = S_1$ be spread out, $\mathbf{E}S < \infty$, $N(u) = \max\{n: S_n \leq u\}$. Assume that process (α_s) has the “lack-of-memory property” that is α_u only depends on $(\eta_s^0)_{S_{N(u)} \leq s \leq u}$.
3. The following analog of Cramér’s condition holds: there exists a real number $\kappa > 0$ such that

$$\mathbf{E} \exp \left(- \int_0^S (\alpha_u - \kappa) du \right) = 1. \tag{3.2}$$

4. There exists $a > \kappa$ such that $\mathbf{E}(e^{aS}) < +\infty$.

Let us discuss the above conditions. Assumption C0.2 that (η_s^0) is a regenerative process is quite natural from a practical point of view. In reliability, the regeneration property is often associated with returning of the process to the state where all components are operating. In accordance with this, the requirement of the lack-of-memory property for (α_s) seems to be natural. It follows that process (α_s) is also regenerative with the same regeneration times S_0, S_1, S_2, \dots . Hypothesis C0.4 requires the existence of an exponential moment for inter-regeneration time S . This is necessary (but not sufficient, in general) for satisfying Cramér’s condition (3.2) which is widely used in various applications since it enables one to simplify studying of underlying models. As we will see, κ is small for highly reliable systems. Therefore, assumption $a > \kappa$ in hypothesis C0.4 is not restrictive in many practical reliability problems. However, condition C0.4 does not seem quite satisfactory, even for Markov processes with a finite number of states. Because of this, we will relax it in some particular cases.

The following theorem contains the desired representation of the limiting value $\lambda(\infty)$.

Theorem 3.2: *Suppose that Conditions C0 hold true. Then*

$$\lambda(\infty) = \frac{\mathbf{E} \int_0^S \alpha_t \exp \left(- \int_0^t (\alpha_u - \kappa) du \right) dt}{\mathbf{E} \int_0^S \exp \left(- \int_0^t (\alpha_u - \kappa) du \right) dt} \tag{3.3}$$

and

$$\lambda(\infty) = \kappa. \tag{3.4}$$

Proof: By Lemma 3.1, the following identity holds for any real κ :

$$\lambda(t) = \frac{\mathbf{E} \left(\alpha_t \exp \left(- \int_0^t (\alpha_u - \kappa) du \right) \right)}{\mathbf{E} \exp \left(- \int_0^t (\alpha_u - \kappa) du \right)} \equiv \frac{N_t}{D_t}. \tag{3.5}$$

Now, take $\kappa > 0$ to satisfy (3.2) and prove that there exist limits for both numerator N_t and denominator D_t . Since (η_s^0) is a zero-delayed regenerative process and the same is true for (α_s) , N_t satisfies the renewal equation

$$N_t = \mathbf{E} \left(\alpha_t \exp \left(- \int_0^t (\alpha_u - \kappa) du \right), S > t \right) + \int_0^t N_{t-u} \mathbf{E} \left(\exp \left(- \int_0^S (\alpha_v - \kappa) dv \right) I(S \in du) \right).$$

Let

$$B(du) = \mathbf{E} \left(\exp \left(- \int_0^S (\alpha_v - \kappa) dv \right) I(s \in du) \right).$$

By (3.2), $B(\cdot)$ is a probability measure. Condition C0.4 implies that

$$b_1 = \int_0^\infty uB(du) < \infty.$$

As S is spread out, the distribution B has the same property. By C0.3 and C0.4,

$$g(t) = \mathbf{E} \left(\alpha_t \exp \left(- \int_0^t (\alpha_u - \kappa) du \right), S > t \right)$$

is bounded, Lebesgue integrable and $\lim_{t \rightarrow \infty} g(t) = 0$. By the key renewal theorem (see Asmussen [1] (Chapter VI, Corollary 1.3)) the following limit exists:

$$N \equiv \lim_{t \rightarrow \infty} N_t = \frac{1}{b_1} \mathbf{E} \int_0^S \alpha_t \exp \left(- \int_0^t (\alpha_u - \kappa) du \right). \quad (3.6)$$

Quite similarly,

$$D \equiv \lim_{t \rightarrow \infty} D_t = \frac{1}{b_1} \mathbf{E} \int_0^S \exp \left(- \int_0^t (\alpha_u - \kappa) du \right). \quad (3.7)$$

Relations (3.5), (3.6), and (3.7) yield (3.3). Now, by (3.2),

$$\mathbf{E} \int_0^S \alpha_t \exp \left(- \int_0^t (\alpha_u - \kappa) du \right) dt = \kappa \mathbf{E} \int_0^S \exp \left(- \int_0^t (\alpha_u - \kappa) du \right) dt. \quad (3.8)$$

Substituting (3.8) into (3.3) we arrive at (3.4). \square

3.2 An upper bound of $\lambda(\infty)$

Theorem 3.2 has the following useful corollaries giving upper bounds of $\lambda(\infty)$ in terms of a stationary expectation of regenerative process (η_s^0) .

Proposition 3.3: *Suppose that Conditions C0 hold true. Then*

$$\lambda(\infty) \leq \lambda^0 \equiv \frac{\mathbf{E} \int_0^S \alpha_u du}{\mathbf{E} S}. \quad (3.9)$$

Proof: Applying the Jensen inequality we have from (3.2) that

$$1 = \mathbf{E} \exp \left(- \int_0^S (\alpha_u - \kappa) du \right) \geq \exp \left(- \mathbf{E} \int_0^S (\alpha_u - \kappa) du \right).$$

Therefore,

$$\mathbf{E} \int_0^S (\kappa - \alpha_u) du \leq 0. \quad (3.10)$$

Since $\mathbf{E} S < \infty$ and $\lambda(\infty) = \kappa$ we arrive at (3.9). \square

Note that the right-hand side of (3.9) is nothing else but the stationary expectation of states of the regenerative process (α_s) .

Remark 3.4: Inequality (3.10) holds under wider assumptions than those listed in Conditions C0. Indeed, it can be used whenever we know that Condition C0.3 holds true, $\lambda(\infty) \leq \kappa$, and $\mathbf{E}S < \infty$.

Remark 3.5: Just to illustrate Remark 3.4, suppose that (η_s) is an irreducible Markov or semi-Markov process on a finite state space E , with, for example, initial state $0 \in E$. Recall that (η_s^0) is supposed to be regenerative, therefore state 0 is recurrent for both processes (η_s^0) and (η_s) . Then they are regenerative with the first regeneration time, respectively, $S = \inf\{s: \eta_s^0 = 0, \eta_{s-}^0 \neq 0\}$ and $\bar{S} = \inf\{s: \eta_s = 0, \eta_{s-} \neq 0\}$. It is easy to verify (using the same method as in the proof of Proposition 2.5) that

$$\mathbf{E} \exp \left(- \int_0^S (\alpha(\eta_u^0) - \kappa) du \right) = \mathbf{E} \left(e^{\kappa \bar{S}} \mathbf{1}_{\{\bar{S} < \sigma\}} \right),$$

and the Cramér condition (3.2) is nothing else but the r -recurrence property (see Coccozza-Thivent and Roussignol [3]) with $r = -\kappa$. Hence, the Cramér condition (3.2) is true if distributions of sojourn time in the states have rational Laplace transforms. Moreover, Theorem 6.3 of Coccozza-Thivent and Roussignol [3] asserts that $\lambda(\infty) = \kappa$ under these conditions. So all hypotheses in Proposition 3.3 are fulfilled (without the necessity to turn to Assumption C0.4).

The following remark gives a useful representation of λ^0 for Markov processes. It is an immediate consequence of definition (3.9) of λ^0 .

Remark 3.6: Let (η_s) be a Markov process conforming to the conditions of Theorem 2.7, C0.1 and C0.2. Then

$$\lambda^0 = \int A(\eta; \mathcal{P}) \pi^0(d\eta), \tag{3.11}$$

π^0 being the stationary distribution of Markov process (η_s^0) .

Since the behavior of process (η_s^0) depends on the behavior of (η_s) until the first hitting time of the subset \mathcal{P} , the estimates contained in Proposition 3.3 and its Corollary 3.6 are attractive for practical use. Intensity λ^0 (see (3.9) to (3.11)) has a clear interpretation and can be calculated in terms of stationary characteristics of the auxiliary process (η_s^0) . We will investigate an accuracy of this approximation in Subsection 5.1.

The following lemma shows that stationary characteristics of (η_s^0) can be obtained in terms of stationary characteristics of process (η_s) provided that (η_s^0) is *time-reversible*.

Given a Markov process on a countable state space E with transition rates matrix A and subset $\mathcal{M} \subset E$, we say that \mathcal{M} is *communicative* if, for any i and j in \mathcal{M} , there exists a path from i to j , i.e., there exists n and i_1, \dots, i_n in \mathcal{M} such that

$$i_1 = i, i_n = j, A(i_m, i_{m+1}) > 0 \quad (1 \leq m \leq n-1).$$

If E is communicative, the process is called *irreducible*. The process is *time reversible* if there exists a probability distribution π such that for any i and j ,

$$\pi(i)A(i, j) = \pi(j)A(j, i). \tag{3.12}$$

It can be easily seen that every distribution π satisfying (3.12) is stationary.

Lemma 3.7: *Let us consider a time-reversible Markov process (η_s) on a discrete state space with stationary probability distribution $\pi(\cdot)$, and let us suppose that set \mathcal{M} is communicative. Then*

$$\pi^0(\eta) = \frac{\pi(\eta)}{\pi(\mathcal{M})}, \quad \eta \in \mathcal{M}.$$

Proof: Let A be the transition matrix (or generator) of the Markov process (η_s) . The

irreducibility and reversibility yield for any i and j (see Disney and Kiessler [8] (Section 2.13.2)) that

$$\begin{aligned}\pi(i) &> 0, \\ A(i, j) &> 0 \Leftrightarrow A(j, i) > 0.\end{aligned}$$

Now, let j_0 be a (fixed) element of \mathcal{M} and for any j in \mathcal{M} , we choose a path from j_0 to j

$$j_0 = i_1, i_2, \dots, i_{n-1}, i_n = j,$$

and define

$$\phi(j) = \frac{A(j_0, i_2)A(i_2, i_3)\dots A(i_{n-1}, j)}{A(i_2, j_0)A(i_3, i_2)\dots A(j, i_{n-1})}.$$

Equation (3.12) implies

$$\pi(j) = \phi(j)\pi(j_0). \quad (3.13)$$

Denote by A_0 the transition matrix of process (η_s^0) . By (2.7), we have

$$A(i, j) = A^0(i, j) \quad (3.14)$$

for all i and j from \mathcal{M} . Let us now define the distribution π^0 on \mathcal{M} by

$$\pi^0(j) = \phi(j)\pi^0(j_0), j \in \mathcal{M}. \quad (3.15)$$

Given j and k from \mathcal{M} such that $A(j, k) > 0$, we have

$$\frac{\pi^0(j)}{\pi^0(k)} = \frac{\phi(j)}{\phi(k)} = \frac{\pi(j)}{\pi(k)} = \frac{A(k, j)}{A(j, k)} = \frac{A^0(k, j)}{A^0(j, k)},$$

(see (3.15), (3.13), (3.12), and (3.14)). Therefore, measure π^0 is a stationary distribution of states of process (η_s^0) . By (3.13) and (3.15), we conclude that measures π and π^0 are proportional on \mathcal{M} , which yields the proof. \square

3.3 The Vesely failure rate

We now derive some useful relations for the Vesely failure rate. Let us start with an ergodic Markov process (η_s) with a discrete state space E and generator A . Here “ergodic” means that the Markov process has a unique stationary probability distribution π such that

$$\lim_{t \rightarrow \infty} \mathbf{E}(f(\eta_t)) = \sum_{\eta \in E} f(\eta)\pi(\eta)$$

for every bounded function f on E . Any irreducible Markov process on a finite state space is ergodic. It follows from (1.3) that the Vesely failure rate has the form

$$\lambda_V(t) = \frac{1}{\mathbf{P}(\eta_t \in \mathcal{M}_b)} \sum_{\eta \in \mathcal{M}_b} \sum_{\xi \in \mathcal{P}} A(\eta, \xi) \mathbf{P}(\eta_t = \eta) = \frac{1}{\mathbf{P}(\eta_t \in \mathcal{M}_b)} \mathbf{E}(\alpha(\eta_t) 1_{\{\eta_t \in \mathcal{M}_b\}}),$$

and consequently, under Condition C(α).3,

$$\lambda_V(\infty) = \lim_{t \rightarrow \infty} \lambda_V(t) = \frac{1}{\pi(\mathcal{M}_b)} \sum_{\eta \in E} \alpha(\eta) 1_{\{\eta \in \mathcal{M}_b\}} \pi(\eta) = \sum_{\eta \in \mathcal{M}_b} \frac{\pi(\eta)}{\pi(\mathcal{M}_b)} A(\eta, \mathcal{P}). \quad (3.16)$$

Proposition 3.8: *Let (η_s) be a time-reversible Markov process on a finite state space with a stationary distribution of states $\pi(\cdot)$ and a bounded generator A ; let set \mathcal{M}_b be commutative. Then,*

$$\lambda_V(\infty) = \lambda^0$$

and

$$\lambda(\infty) \leq \lambda_V(\infty).$$

Proof: The proof follows from Example 2.3, Remark 3.6, Lemma 3.7, and formula (3.16) by using the result indicated in Remark 3.5. \square

Remark 3.9: Engineers are often interested in the Vesely failure rate since it can easily be computed. Such calculations can be done with the help of a fault tree when the system consists of independent components (see Example 3.14). In such a case the system behavior can be described by a finite reversible Markov process, if the failure and repair rates of each component are constant and strictly positive. Engineers observed (from practical examples) that $\lambda(\infty) \leq \lambda_V(\infty)$ but, as far as we know, this result had not been proved. Now it is done and it is an important result for applications.

As we will see in Proposition 3.15, the equality $\lambda^0 = \lambda_V(\infty)$ is still true for a general redundant system introduced in Example 2.6 (see also Example 3.14) in the case of constant failure rates and general repair rates.

We now come back to general Markov processes. The following proposition generalizes representation (3.16) of general Markov processes having the Feller property.

Proposition 3.10: *Let (η_s) be an ergodic Feller process satisfying assumptions of Theorem 2.7 with stationary distribution of states π . Then*

$$\lambda_V(\infty) = \frac{1}{\pi(\mathcal{M}_b)} \int_{\mathcal{M}_b} A(\eta; \mathfrak{P}) \pi(d\eta). \tag{3.17}$$

Proof: By Remark 2.8, function $f(\eta) = 1_{\{\eta \in \mathfrak{P}\}}$ belongs to the domain of the generator of (η_s) . The generator's definition and (1.3) yield

$$\begin{aligned} \lambda_V(t) &= \frac{1}{\mathbf{P}(\eta_t \in \mathcal{M}_b)} \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \int_{\mathcal{M}_b} \mathbf{P}(\eta_{t+\Delta} \in \mathfrak{P} \mid \eta_t = \eta) \mathbf{P}(\eta_t \in d\eta) \\ &= \frac{1}{\mathbf{P}(\eta_t \in \mathcal{M}_b)} \lim_{\Delta \rightarrow 0} \int_{\mathcal{M}_b} \frac{1}{\Delta} (\mathbf{E}(f(\eta_{t+\Delta}) \mid \eta_t = \eta) - f(\eta)) \mathbf{P}(\eta_t \in d\eta) \\ &= \frac{1}{\mathbf{P}(\eta_t \in \mathcal{M}_b)} \mathbf{E}A(\eta_t; \mathfrak{P}). \end{aligned}$$

As the process is ergodic, this yields the desired result.

Example 3.11: Semi-Markov process on a discrete state space. We return to Examples 2.4 and 2.10 and preserve corresponding notations. Let the semi-Markov process be ergodic and let π be its stationary distribution of states. Example 2.10 and Proposition 3.10 yield

$$\lambda_V(\infty) = \frac{1}{\pi'(\mathcal{M}_b')} \sum_{i \in \mathcal{M}_b} \int_0^\infty \sum_{j \in \mathfrak{P}} A(i, j, y) \pi'(i, dy), \tag{3.18}$$

where π' is the stationary distribution of Markov process (η_s, y_s) . It is not difficult to prove (see, for example, Disney and Kiessler [8] (Section 2.7)) that

$$\pi'(i, dy) = K\nu(i) \exp \left(- \int_0^y h(i, s) ds \right) dy, \tag{3.19}$$

where function h is defined in Example 2.4, $\nu(\cdot)$ is the stationary distribution of the embedded Markov chain X with transition probabilities $\mathbf{p}(i, j) = \int q(i, j, y) dy$, and K is a norming constant. Let us denote by π the stationary distribution of the semi-Markov process. It follows from (3.19) that

$$\pi(i) = \int_0^\infty \pi'(i, dy) = K\nu(i) \int_0^\infty \exp \left(- \int_0^y h(i, s) ds \right) dy = K\nu(i)m(i),$$

where the mean sojourn time in the state i , $m(i) = \mathbf{E}(T_1 | \eta_0 = i)$, is finite. Thus, π can be viewed as the stationary distribution of states of a Markov process with the same embedded Markov chain X and the same mean sojourn times $m(i)$. This Markov process has the generator

$$\mathbf{a}(i, j) = \frac{\mathbf{p}(i, j)}{m(i)}.$$

Substituting (3.19) into (3.18) and applying (2.9) to (2.10), we get

$$\begin{aligned} \lambda_V(\infty) &= \frac{1}{\pi(\mathcal{M}_b)} \sum_{i \in \mathcal{M}_b} \sum_{j \in \mathfrak{P}} \int_0^\infty p(i, j, y) h(i, y) K\nu(i) \exp\left(-\int_0^y h(i, s) ds\right) dy \\ &= \frac{1}{\pi(\mathcal{M}_b)} \sum_{i \in \mathcal{M}_b} \sum_{j \in \mathfrak{P}} K\nu(i) \int_0^\infty q(i, j, y) dy \\ &= \frac{1}{\pi(\mathcal{M}_b)} \sum_{i \in \mathcal{M}_b} \sum_{j \in \mathfrak{P}} K\nu(i) \mathbf{p}(i, j) \\ &= \frac{1}{\sum_{k \in \mathcal{M}_b} \nu(k) m(k)} \sum_{i \in \mathcal{M}_b} \sum_{j \in \mathfrak{P}} \nu(i) \mathbf{p}(i, j). \end{aligned}$$

This yields

$$\lambda_V(\infty) = \sum_{i \in \mathcal{M}_b} \frac{\pi(i)}{\pi(\mathcal{M}_b)} \sum_{j \in \mathfrak{P}} \frac{\mathbf{p}(i, j)}{m(i)} = \sum_{i \in \mathcal{M}_b} \frac{\pi(i)}{\pi(\mathcal{M}_b)} \sum_{j \in \mathfrak{P}} \mathbf{a}(i, j). \tag{3.20}$$

The arguments above imply the following result.

Proposition 3.12: *For an ergodic semi-Markov process on a discrete state space, the asymptotic Vesely failure rate is equal to the asymptotic Vesely failure rate of a Markov process having the same embedded Markov chain and the same mean sojourn times in all states.*

Remark 3.13: It can be seen from (3.20) that, in the Markovian case, $\lambda_V(\infty) = 1/MUT$, where MUT means Mean Up Time of the system (see Pagès and Gondran [20]). The last proposition shows that this is also true in the semi-Markovian case.

Example 3.14: Redundant system. Consider the redundant system described in Examples 2.6 and 2.11 and preserve the same notations. Let us embed (η_s) into the Markov process (η_s, y_s) . Let π' be its stationary distribution of states. By Proposition 3.10 and equation (2.32)

$$\lambda_V(\infty) = \frac{1}{\pi'(\mathcal{M}_b')} \sum_{\eta \in \mathcal{M}_b} \int_0^\infty \sum_{j \in R(\eta)} a^{(i)}(\eta, y) \pi'(\eta, dy). \tag{3.21}$$

Since all components of the system are independent, the measure π' has the product form

$$\pi'(\eta, y) = \prod_{i=1}^N \pi^{(i)}(\eta(i), y(i)),$$

and the same is true for the stationary distribution π of states of process (η_s)

$$\pi(\eta) = \prod_{i=1}^N \pi^{(i)}(\eta(i)).$$

For any i , function $a^{(i)}$ depends on $\eta(i)$ and $y(i)$ only. Hence, we can write $a^{(i)}(\eta(i), y(i))$ instead of $a^{(i)}(\eta, y)$. Let γ_i and δ_i be the mean sojourn time of the i th component in “good” state 1 and in “bad” state 0, respectively, and

$$m^{(i)}(\eta(i)) = \begin{cases} \gamma_i, & \text{if } \eta(i) = 1, \\ \delta_i, & \text{if } \eta(i) = 0. \end{cases}$$

It can be easily seen from (3.19) that

$$\pi^{(i)}(\eta(i), y(i)) = \frac{1}{\gamma_i + \delta_i} \exp \left(- \int_0^{y(i)} a^{(i)}(\eta(i), s) ds \right),$$

and, consequently,

$$\int_0^\infty a^{(i)}(\eta, y) \pi'(\eta, dy) = \frac{1}{\gamma_i + \delta_i} \prod_{j: j \neq i} \frac{m^{(j)}(\eta(j))}{\gamma_j + \delta_j} = \frac{1}{\gamma_i + \delta_i} \prod_{j: j \neq i} \pi^{(j)}(\eta(j)).$$

Assume additionally that the system is *coherent*. This means that if the system is operating (resp., failed) and if one more component becomes operating (resp., failed) then the system remains operating (resp., failed). This assumption is typical and natural for reliability studies. Then

$$\begin{aligned} & \sum_{\eta \in \mathcal{M}} \sum_{i \in R(\eta)} \int_0^\infty a^{(i)}(\eta, y) \pi'(\eta, dy) \\ &= \sum_i \frac{1}{\gamma_i + \delta_i} \sum_{\eta: i \in R(\eta)} \prod_{j: j \neq i} \pi^{(j)}(\eta(j)) \\ &= \sum_i \frac{1}{\gamma_i + \delta_i} (\pi(\eta: (\eta(1), \dots, \eta(i-1), 0, \eta(i+1), \dots, \eta(N)) \in \mathfrak{P}) \\ & \quad - \pi(\eta: (\eta(1), \dots, \eta(i-1), 1, \eta(i+1), \dots, \eta(N)) \in \mathfrak{P})). \end{aligned} \tag{3.22}$$

Equations (3.21) and (3.22) yield the following formula, well-known to engineers:

$$\begin{aligned} \lambda_V(\infty) &= \frac{1}{\pi(\mathcal{M})} \sum_i \frac{1}{\gamma_i + \delta_i} (\pi(\eta: (\eta(1), \dots, \eta(i-1), 0, \eta(i+1), \dots, \eta(N)) \in \mathfrak{P}) \\ & \quad - \pi(\eta: (\eta(1), \dots, \eta(i-1), 1, \eta(i+1), \dots, \eta(N)) \in \mathfrak{P})). \end{aligned}$$

It can easily be computed with the help of fault trees and existing software.

The following result generalizes Remark 3.9. It uses PH-distributions (see Asmussen [1] (Section II.6) and Kalashnikov [15] (Subsection 9.5.3)) which can be treated as distributions of absorption times for Markov processes with a finite state space.

Proposition 3.15: *Consider the redundant system described in Example 2.6. Let the system be coherent with constant failure rates and general strictly positive repair rates, and repair times having finite means. Then we have:*

$$\lambda^0 = \lambda_V(\infty).$$

Consequently, if the repair times have PH-distributions or if conditions C0.3 and C0.4 are satisfied, then

$$\lambda(\infty) \leq \lambda_V(\infty).$$

Proof: We use notations from Examples 2.6 and 2.11. Since failure rates are constant, it is not necessary to consider all $y(i)$'s from (2.16) to obtain a Markov process. Define another vector \underline{y} with N components:

$$\underline{y}(i) = \begin{cases} y(i), & \text{if } \eta(i) = 0, \\ 0, & \text{if } \eta(i) = 1. \end{cases}$$

Process $(\eta_s, \underline{y}_s)$ is Markov with generator

$$\underline{A}f(\eta, \underline{y}) = \sum_{i: \eta(i)=0} \frac{\partial f(\eta, \underline{y})}{\partial y(i)} + \sum_i a^{(i)}(\eta(i), \underline{y}(i))(f(\eta^{(i)}, \underline{y}^{(i)}) - f(\eta, \underline{y})),$$

where

$$a^{(i)}(\eta(i), \underline{y}(i)) = \begin{cases} \mu^{(i)}(\underline{y}(i)), & \text{if } \eta(i) = 0, \\ \lambda^{(i)}(\underline{y}(i)) = \lambda^{(i)}(\underline{0}) = \lambda^{(i)}, & \text{if } \eta(i) = 1. \end{cases}$$

All assumptions of Theorem 2.7 are fulfilled and thus

$$\underline{A}^0 f(\eta, \underline{y}) = \sum_{i: \eta(i)=0} \frac{\partial f(\eta, \underline{y})}{\partial y(i)} + \sum_{i \notin R(\eta)} a^{(i)}(\eta(i), \underline{y}(i))(f(\eta^{(i)}, \underline{y}^{(i)}) - f(\eta, \underline{y})).$$

Let us prove that stationary distribution $\underline{\pi}^0$ of the Markov process with generator \underline{A}^0 is defined by the relation

$$\underline{\pi}^0(\varphi) = K_0 \sum_{\eta \in \mathcal{M}} \int \varphi(\eta, \underline{y}) \prod_{j: \eta(j)=0} \exp\left(-\int_0^{\underline{y}(j)} \mu^{(j)}(s) ds\right) \prod_{j: \eta(j)=1} \frac{1}{\lambda^{(j)}} \prod_{j: \eta(j)=0} d\underline{y}(j), \quad (3.23)$$

where φ is a bounded measurable function and $K_0 > 0$ is a norming constant. Define functions g_j and g as

$$g_j(\eta(j), \underline{y}(j)) = \begin{cases} \exp\left(-\int_0^{\underline{y}(j)} \mu^{(j)}(s) ds\right), & \text{if } \eta(j) = 0, \\ 1/\lambda^{(j)}, & \text{if } \eta(j) = 1, \end{cases} \quad (3.24)$$

$$g(\eta, \underline{y}) = \prod_j g_j(\eta(j), \underline{y}(j)), \quad (3.25)$$

and prove that, for every function f from the domain of \underline{A}^0 ,

$$C(f) \equiv \sum_{\eta \in \mathcal{M}} \int \underline{A}^0 f(\eta, \underline{y}) g(\eta, \underline{y}) \prod_{j: \eta(j)=0} d\underline{y}(j) = 0.$$

Let i be such that $\eta(i) = 1$. Then by integrating by parts,

$$\begin{aligned} & \int \frac{\partial f(\eta, \underline{y})}{\partial y(i)} g(\eta, \underline{y}) \prod_{j: \eta(j)=0} d\underline{y}(j) \\ &= \int \prod_{j: j \neq i} g_j(\eta(j), \underline{y}(j)) \prod_{\substack{j: j \neq i, \\ \eta(j)=0}} d\underline{y}(j) \int_0^\infty \frac{\partial f(\eta, \underline{y})}{\partial y(i)} \exp\left(-\int_0^{\underline{y}(i)} \mu^{(i)}(s) ds\right) d\underline{y}(i) \\ &= - \int f(\eta, \underline{y}^{(i)}) \prod_{j: j \neq i} g_j(\eta(j), \underline{y}(j)) \prod_{\substack{j: j \neq i, \\ \eta(j)=0}} d\underline{y}(j) + \int \mu^{(i)}(\underline{y}(i)) f(\eta, \underline{y}) g(\eta, \underline{y}) \prod_{\substack{j: j \neq i, \\ \eta(j)=0}} d\underline{y}(j). \end{aligned}$$

Using the fact that $\underline{y} = \underline{y}^{(i)}$ if $\eta(i) = 1$ and system's coherence, we have

$$C(f) = - \sum_{\eta \in \mathcal{M}} \sum_{i: \eta(i)=0} \int f(\eta, \underline{y}^{(i)}) \prod_{j: j \neq i} g_j(\eta(j), \underline{y}(j)) \prod_{\substack{j: j \neq i, \\ \eta(j)=0}} d\underline{y}(j)$$

$$\begin{aligned}
 & + \sum_{\eta \in \mathcal{M}_b} \sum_{i: \eta^{(i)} = 0} \int \mu^{(i)}(\underline{y}^{(i)}) f(\eta, \underline{y}) g(\eta, \underline{y}) \prod_{j: \eta^{(j)} = 0} d\underline{y}^{(j)} \\
 & + \sum_{\eta \in \mathcal{M}_b} \sum_{i: \eta^{(i)} = 0} \int f(\eta^{(i)}, \underline{y}^{(i)}) \prod_{j: j \neq i} g_j(\eta^{(j)}, \underline{y}^{(j)}) \prod_{\substack{j: j \neq i, \\ \eta^{(j)} = 0}} d\underline{y}^{(j)} \\
 & - \sum_{\eta \in \mathcal{M}_b} \sum_{i: \eta^{(i)} = 0} \int \mu^{(i)}(\underline{y}^{(i)}) f(\eta, \underline{y}) f(\eta, \underline{y}) g(\eta, \underline{y}) \prod_{j: \eta^{(j)} = 0} d\underline{y}^{(j)} \\
 & + \sum_{\eta \in \mathcal{M}_b} \sum_{\substack{i: \eta^{(i)} \in \mathcal{M}_b, \\ \eta^{(i)} = 1}} \int f(\eta^{(i)}, \underline{y}^{(i)}) \prod_{j: j \neq i} g_j(\eta^{(j)}, \underline{y}^{(j)}) \prod_{\substack{j: j \neq i, \\ \eta^{(j)} = 0}} d\underline{y}^{(j)} \\
 & - \sum_{\eta \in \mathcal{M}_b} \sum_{\substack{i: \eta^{(i)} \in \mathcal{M}_b, \\ \eta^{(i)} = 1}} \int f(\eta, \underline{y}^{(i)}) \prod_{j: j \neq i} g_j(\eta^{(j)}, \underline{y}^{(j)}) \prod_{\substack{j: j \neq i, \\ \eta^{(j)} = 0}} d\underline{y}^{(j)} \\
 & = - \sum_{\eta \in \mathcal{M}_b} \sum_{i: \eta^{(i)} \in \mathcal{M}_b} \int f(\eta, \underline{y}^{(i)}) \prod_{j: j \neq i} g_j(\eta^{(j)}, \underline{y}^{(j)}) \prod_{\substack{j: j \neq i, \\ \eta^{(j)} = 0}} d\underline{y}^{(j)} \\
 & + \sum_{\eta \in \mathcal{M}_b} \sum_{i: \eta^{(i)} \in \mathcal{M}_b} \int f(\eta^{(i)}, \underline{y}^{(i)}) \prod_{j: j \neq i} g_j(\eta^{(j)}, \underline{y}^{(j)}) \prod_{\substack{j: j \neq i, \\ \eta^{(j)} = 0}} d\underline{y}^{(j)} \\
 & = 0.
 \end{aligned}$$

Similar calculations show that the stationary probability distribution for process $(\eta_s, \underline{y}_s)$ is given by the relation

$$\pi(\varphi) = K \sum_{\eta \in E} \int \varphi(\eta, \underline{y}) g(\eta, \underline{y}) \prod_{j: \eta^{(j)} = 0} d\underline{y}^{(j)}, \tag{3.26}$$

where φ is a bounded measurable function and $K > 0$ is a norming constant.

Equations (3.23), (3.24), (3.25), and (3.26) imply that

$$\pi^0(\cdot) = \frac{\pi(\cdot)}{\pi(\mathcal{M}_b \times \mathbf{R}_+^N)}.$$

Theorem 2.7 yields $A((\eta, \underline{y}); \mathfrak{P}) = \sum_{i \in R(\eta)} \lambda^{(i)} = \alpha(\eta)$. Then the statement of the present proposition is a consequence of Proposition 3.10, Remarks 3.5, 3.6, and Theorem 3.2. \square

Remark 3.16: Conditions of Proposition 3.15 are optimal in the following sense. Generally, $\lambda_V(\infty) \neq \lambda^0$ for a redundant system where failure rates are not constant. Moreover, the inequality $\lambda_V(\infty) < \lambda(\infty)$ may hold true. As an example, let us consider two independent components C1 and C2 in parallel. Suppose that the lifetime of component C1 has the Erlang distribution with parameters 2 and 1 (that is the lifetime can be viewed as a sum of two independent random variables, each one being exponentially distributed with parameter 1) and its repair rate is $\mu^{(1)} = 10$. Suppose that the failure rate of component C2 is $\lambda^{(2)} = 1$ and its repair rate is $\mu^{(2)} = 10$. It can be seen that

$$\lambda(\infty) = 8.7024 \cdot 10^{-2}, \quad \lambda_V(\infty) = 8.6957 \cdot 10^{-2}, \quad \lambda^0 = 9.4340 \cdot 10^{-2},$$

that is, $\lambda^0 \neq \lambda_V(\infty)$ and $\lambda_V(\infty) < \lambda(\infty)$.

4. A Stationary Representation of the Asymptotic Failure Rate

Throughout the remainder of this paper we will assume that the following Conditions C1 are met:

Conditions C1

1. Conditions C0 hold.
2. Intensity α_u depends on the current state η_u only, that is $\alpha_u = \alpha(\eta_u^0)$.

In this section, we construct a regenerative process $(\tilde{\eta}_s)$ such that $\lambda(\infty)$ is equal to the stationary mean value of $\alpha(\tilde{\eta}_s)$. For this, we define the following measure on subsets of \mathcal{M} :

$$\tilde{\pi}(\cdot) = \frac{\mathbf{E} \int_0^S I(\eta_t^0 \in \cdot) \exp\left(-\int_0^t (\alpha_u - \kappa) du\right) dt}{\mathbf{E} \int_0^S \exp\left(-\int_0^t (\alpha_u - \kappa) du\right) dt}. \tag{4.1}$$

Then, by (3.3), $\lambda(\infty)$ can be represented as

$$\lambda(\infty) = \int_{\mathcal{M}} \alpha(\eta) \tilde{\pi}(d\eta). \tag{4.2}$$

Remark 4.1: It can be easily seen (using (2.2) and the same arguments as in the proof of Theorem 3.2) that $\tilde{\pi}$ is the quasi-stationary distribution of process (η_s) , i.e.,

$$\tilde{\pi}(\cdot) = \lim_{t \rightarrow \infty} \mathbf{P}(\eta_t \in \cdot \mid \sigma > t).$$

Before constructing a regenerative process with stationary distribution $\tilde{\pi}$, let us build a regenerative process $\eta^1 \equiv (\eta_s^1)$ with initial distribution $\tilde{\pi}$.

Formula (4.1) yields the following general construction. Let us regard random process $\eta^0 \equiv (\eta_s^0)(\omega)$ as defined on probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and expand this space as follows (a similar construction was used by Thorisson [22]; see also Kalashnikov [16] (Section 1.6)). Let \mathcal{B} be the collection of all Borel subsets of $[0, 1]$. Set

$$\omega' = (\omega, u), \quad \omega \in \Omega, u \in [0, 1]; \tag{4.3}$$

$$\Omega' = \Omega \times [0, 1]; \quad \mathcal{F}' = \mathcal{F} \times \mathcal{B};$$

$$\mathbf{P}'(d\omega') = \mathbf{P}'(d\omega, du) = \mathbf{P}(d\omega)du; \tag{4.4}$$

$$\mathbf{P}''(d\omega') = \frac{1}{D} S(\omega) \exp\left(-\int_0^{uS(\omega)} (\alpha(\eta_v^0(\omega)) - \kappa) dv\right) \mathbf{P}'(d\omega'), \tag{4.5}$$

where

$$D = \mathbf{E} \int_0^S \exp\left(-\int_0^t (\alpha(\eta_v^0) - \kappa) dv\right) dt. \tag{4.6}$$

This construction has the following meaning. Formulas (4.3) and (4.4) imply that the r.v. U is defined independently of other random elements (this leads to the probability \mathbf{P}'). Then we introduce a new probability measure \mathbf{P}'' on the expanded probability space (see (4.5) and (4.6)), where D is a norming constant. After this, we can define a new process $\eta^1 \equiv (\eta_s^1)$ on the probability space $(\Omega', \mathcal{F}', \mathbf{P}'')$ by the relation

$$\eta_t^1(\omega') = \eta_{t+uS(\omega)}^0(\omega) \equiv (\theta_{uS(\omega)} \eta^0)_t, \quad \omega' = (\omega, u), \tag{4.7}$$

where θ_s stands for the shift operator on time s onward. We are going to see that construction

(4.3) through (4.7) implies that process η^1 can be viewed as regenerative with regeneration times

$$S_k^1(\omega') = S_{k+1}^0(\omega) - S(\omega)u, \quad k \geq 0. \tag{4.8}$$

Consider the pair

$$(\eta^1, S^1)(\omega') = ((\eta_s^1), S_0^1, S_1^1, \dots)(\omega'),$$

defined by (4.7) and (4.8). Then the following chain of relations holds

$$\begin{aligned} & \mathbf{P}''(\theta_{S_k^1}(\eta^1, S^1)(\omega') \in \cdot) \\ &= \frac{1}{D} \int_{\Omega} S(\omega) \int_0^1 I(\theta_{S_{k+1}^0(\omega) - uS(\omega)} \theta_{uS(\omega)}(\eta^0, S^0)(\omega) \in \cdot) \\ & \quad \times \exp\left(-\int_0^{uS(\omega)} (\alpha(\eta_v^0) - \kappa)dv\right) du \mathbf{P}(d\omega) \\ &= \frac{1}{D} \int_{\Omega} I(\theta_{S_{k+1}^0(\omega)}(\eta^0, S^0)(\omega) \in \cdot) \int_0^{S(\omega)} \exp\left(-\int_0^t (\alpha(\eta_v^0) - \kappa)dv\right) dt \mathbf{P}(d\omega) \\ &= \mathbf{E}I(\theta_{S_{k+1}^0(\omega)}(\eta^0, S^0)(\omega) \in \cdot) \frac{1}{D} \mathbf{E} \int_0^{S(\omega)} \exp\left(-\int_0^t (\alpha(\eta_v^0) - \kappa)dv\right) dt \\ &= \mathbf{E}I(\theta_{S_{k+1}^0(\omega)}(\eta^0, S^0)(\omega) \in \cdot) \\ &= \mathbf{E}I((\eta^0, S^0)(\omega) \in \cdot). \end{aligned} \tag{4.9}$$

Therefore, all shifts $\theta_{S_k^1}(\eta^1, S^1)$ are identically distributed. Quite similarly, one can prove that events

$$\{\theta_{S_k^1}(\eta^1, S^1) \in \cdot\}$$

and

$$\{(S_0^1, \dots, S_k^1) \in \cdot\}$$

are independent. Moreover, it follows from (4.1), (4.5), and (4.7) that the *initial distribution* of process η^1 is $\tilde{\pi}$.

Remark 4.2: Equality (4.9) shows that all shifts $\theta_{S_k^1}(\eta^1, S^1)$ are distributed as (η^0, S^0) . By the terminology adopted in the theory of regenerative processes, this means that (η^1, S^1) is a *version* of (η^0, S^0) , i.e., these two processes have identically distributed cycles but perhaps different delays. Regenerative process (η^1, S^1) is *delayed*, in general; that is, $\mathbf{P}(S_0^1 \neq 0) > 0$.

Remark 4.3: If η^0 is a Markov process, then the above construction shows that η^1 is also a Markov process with the same transition probabilities as η^0 (and with the *initial distribution* $\tilde{\pi}$).

We now construct a regenerative process $\tilde{\eta}$ that has $\tilde{\pi}$ as its stationary distribution. For this, let us introduce processes $\eta^i, i \geq 1$ and positive random variables T^i such that

- processes $\eta^i, i \geq 0$ are independent;
- process η^0 is a probabilistic replica of the “original” process η^0 in the sense that it has the same probability law (because of this, we did not change its notation);
- processes $\eta^i, i \geq 1$ are i.i.d. probabilistic replicas of the “original” process η^1 ;
- for any $i \geq 0$,

$$\mathbf{P}(T^i \leq t \mid (\eta_s^i)_{s \geq 0}) = \mathbf{P}(T^i \leq t \mid (\eta_s^i)_{0 \leq s \leq t}) = 1 - \exp \left(- \int_0^t \alpha(\eta_u^i) du \right). \quad (4.10)$$

Let us denote by $(S_k^i)_{k \geq 0}$ regeneration times of process η^i . Recall that $S_0^i \neq 0$ in general (see Remark 4.2).

Hereafter, let us regard all processes as defined on the same measurable space (Ω', \mathcal{F}') and preserve notations \mathbf{P} , \mathbf{P}' , and \mathbf{P}'' for the measures indicated in (4.4) and (4.5). Construct process $\tilde{\eta}$ from fragments of the regenerative process η^0 and its i.i.d. versions $\eta^i, i \geq 1$, as follows:

$$\tilde{\eta}_t(\omega') = \begin{cases} \eta_t^0(\omega), & \text{if } t < T^0, \\ \eta_{t - (T^0 T + \dots + T^{i-1})}^i(\omega'), & \text{if } T^0 + \dots + T^{i-1} \leq t < T^0 + \dots + T^i, i \geq 1. \end{cases} \quad (4.11)$$

Then for each $i \geq 1$, set $\tilde{S}^i = \{T^- + \dots + T^{i-1} + S_k^i : k \geq 0, 0 \leq S_k^i < T^i\}$ is the collection of regeneration times of the process $\eta_{t - T^0 - \dots - T^{i-1}}^i$ occurring before its stopping at time T^i . Set \tilde{S}^i may be empty if $T^i \leq S_0^i$. Process $\tilde{\eta}$ thus defined can be viewed as regenerative in at least two different ways:

- (i) with regeneration times $(T^0 + \dots + T^k)_{k \geq 0}$;
- (ii) with regeneration times $\tilde{S} = (\tilde{S}^0, \tilde{S}^1, \tilde{S}^2, \dots) = (\tilde{S}_0, \tilde{S}_1, \dots)$ consisting of all ‘‘occurred’’ regeneration times. By construction, $\tilde{S}_0 = 0$.

Let us call the regeneration times indicated above as those of the *first* and *second* types, respectively.

Theorem 4.4: *Under assumptions C1, there exists a regenerative process $\tilde{\eta}$ with stationary distribution $\tilde{\pi}$ such that*

$$\lambda(\infty) \int_{\mathcal{M}} \alpha(\eta) \tilde{\pi}(d\eta).$$

Moreover,

$$\tilde{\eta}_t = \eta_t^0 \text{ if } t < T^0, \quad \mathbf{P}(\tilde{\eta}_{T^0} \in \cdot) = \tilde{\pi}(\cdot),$$

where T^0 is an r.v. such that

$$\mathbf{P}(T^0 \leq t \mid (\eta_s^0)_{s \geq 0}) = \mathbf{P}(T^0 \leq t \mid (\eta_s^0)_{0 \leq s \leq t}) = 1 - \exp \left(- \int_0^t \alpha(\eta_u^0) du \right).$$

Proof: Let us consider the regenerative process $(\tilde{\eta}, \tilde{S})$. In order not to introduce new notation, we denote by $\tilde{\pi}$ the stationary distribution of $\tilde{\eta}$ which can be written as

$$\tilde{\pi}(\cdot) = \frac{1}{\mathbf{E}\tilde{S}_1} \mathbf{E} \int_0^{\tilde{S}_1} I(\tilde{\eta}_t \in \cdot) dt. \quad (4.12)$$

To prove that distributions (4.1) and (4.12) are the same, it is sufficient to state that

$$N \equiv \mathbf{E} \int_0^{\tilde{S}_1} I(\tilde{\eta}_t \in \cdot) dt = \mathbf{E} \int_0^S I(\eta_t^0 \in \cdot) \exp \left(- \int_0^t (\alpha(\eta_u^0) - \kappa) du \right) dt. \quad (4.13)$$

By construction (4.11),

$$\begin{aligned}
 N &= \mathbf{E} \int_0^{\tilde{S}_1 \wedge T^0} I(\tilde{\eta}_t \in \cdot) dt + \mathbf{E} \left(I(T^0 < \tilde{S}_1) \int_{T^0}^{\tilde{S}_1} I(\tilde{\eta}_t \in \cdot) dt \right) \\
 &= \mathbf{E} \int_0^{S_1^0} I(\eta_t^0 \in \cdot) \exp \left(- \int_0^t \alpha(\eta_u^0) du \right) dt + \mathbf{P}(T^0 < \tilde{S}_1) N_1,
 \end{aligned} \tag{4.14}$$

where

$$N_1 = \mathbf{E} \int_0^{S_1'} I(\theta_{T^0} \tilde{\eta}_t \in \cdot) dt$$

and S_1' is the first regeneration time of the second type for the shifted process $\theta_{T^0} \tilde{\eta}$; evidently, $S_1' > T^0$. Since S_1^0 is measurable with respect to (η_s^0) , it can be regarded as a constant when conditioning with respect to (η_s^0) . Therefore, equality (4.10) yields

$$\mathbf{P}(T^0 < \tilde{S}_1) = \mathbf{P}(T^0 < S_1^0) = \left(1 - \mathbf{E} \exp \left(- \int_0^{S_1^0} \alpha(\eta_u^0) du \right) \right). \tag{4.15}$$

All shifts $\theta_{T^0 + \dots + T^k} \tilde{\eta}$ are identically distributed. Because of this we have, similarly to (4.14),

$$N_1 = \mathbf{E}'' \int_0^{S_0^1} I(\eta_t^1 \in \cdot) \exp \left(- \int_0^t \alpha(\eta_v^1) dv \right) dt + \left(1 - \mathbf{E}'' \exp \left(- \int_0^{S_0^1} \alpha(\eta_t^1) dt \right) \right) N_1.$$

Therefore,

$$N_1 = \frac{\mathbf{E}'' \int_0^{S_0^1} I(\eta_t^1 \in \cdot) \exp \left(- \int_0^t \alpha(\eta_v^1) dv \right) dt}{\mathbf{E}'' \exp \left(- \int_0^{S_0^1} \alpha(\eta_t^1) dt \right)}. \tag{4.16}$$

Let us use construction (4.7) of process η^1 in order to find N_1 . By (4.3) to (4.8) and changing variables, we have

$$\begin{aligned}
 &\mathbf{E}'' \int_0^{S_0^1} I(\eta_t^1 \in \cdot) \exp \left(- \int_0^t \alpha(\eta_v^1) dv \right) dt \\
 &= \frac{1}{D} \int_{\Omega} S(\omega) \int_0^1 \exp \left(- \int_0^{uS(\omega)} (\alpha(\eta_v^0(\omega)) - \kappa) dv \right) \\
 &\quad \times \int_0^{S(\omega)(1-u)} I(\eta_{t+uS(\omega)}^0 \in \cdot) \exp \left(- \int_0^t \alpha(\eta_{v+uS(\omega)}^0(\omega)) dv \right) dt du \mathbf{P}(d\omega) \\
 &= \frac{1}{D} \mathbf{E} \int_0^S \exp \left(- \int_0^s (\alpha(\eta_v^0) - \kappa) dv \right) \int_0^{S-s} I(\eta_{t+s}^0 \in \cdot) \exp \left(- \int_0^t \alpha(\eta_{v+s}^0) dv \right) dt ds
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{D} \mathbf{E} \int_0^S I(\eta_u^0 \in \cdot) \exp \left(- \int_0^u \alpha(\eta_v^0) dv \right) \int_0^u e^{\kappa s} ds du \\
 &= \frac{1}{\kappa D} \mathbf{E} \int_0^S I(\eta_u^0 \in \cdot) \exp \left(- \int_0^u \alpha(\eta_v^0) dv \right) (e^{\kappa u} - 1) du.
 \end{aligned} \tag{4.17}$$

Similarly, by (3.2),

$$\begin{aligned}
 \mathbf{E}'' \exp \left(- \int_0^{S_0^1} \alpha(\eta_t^1) dt \right) &= \frac{1}{\kappa D} \mathbf{E} \exp \left(- \int_0^S \alpha(\eta_v^0) dv \right) (e^{\kappa S} - 1) \\
 &= \frac{1}{\kappa D} \left(1 - \mathbf{E} \exp \left(- \int_0^S \alpha(\eta_v^0) dv \right) \right).
 \end{aligned} \tag{4.18}$$

Substituting this into (4.16), we have

$$N_1 = \frac{\mathbf{E} \int_0^S I(\eta_u^0 \in \cdot) \exp(-\int_0^u \alpha(\eta_v^0) dv) (e^{\kappa u} - 1) du}{1 - \mathbf{E} \exp(-\int_0^S \alpha(\eta_v^0) dv)}. \tag{4.19}$$

It follows from (4.14), (4.15), and (4.19) that

$$N = \mathbf{E} \int_0^S I(\eta_t^0 \in \cdot) \exp \left(- \int_0^t (\alpha(\eta_u^0) - \kappa) du \right) dt.$$

Hence, relation (4.13) holds true and Theorem 4.4 is proved. □

Remark 4.5: The construction above is a generalization of the construction from Cocozza-Thivent and Roussignol [4] proposed there for Markov processes on a discrete state space. Let us notice that Conditions C1 are met for such processes (see Remark 3.5) except for Assumption C0.4 which is not needed.

It follows from Theorem 4.4 that

$$\lambda(\infty) = \frac{1}{\mathbf{E} \tilde{S}_1} \mathbf{E} \int_0^{\tilde{S}_1} \alpha(\tilde{\eta}_t) dt. \tag{4.20}$$

Now, let us give another expression for $\lambda(\infty)$ in terms of processes η^0 and η^1 . Similarly to (4.14) and (4.16),

$$\begin{aligned}
 \mathbf{E} \int_0^{\tilde{S}_1} \alpha(\tilde{\eta}_t) dt &= \mathbf{E} \int_0^S \alpha(\eta_t^0) \exp \left(- \int_0^t \alpha(\eta_u^0) du \right) dt \\
 &+ \left(1 - \mathbf{E} \exp \left(- \int_0^S \alpha(\eta_t^0) dt \right) \right) \frac{\mathbf{E}'' \int_0^{S_0^1} \alpha(\eta_t^1) \exp \left(- \int_0^t \alpha(\eta_u^1) du \right) dt}{\mathbf{E}'' \exp \left(- \int_0^{S_0^1} \alpha(\eta_t^1) dt \right)} \\
 &= \frac{1 - \mathbf{E} \exp \left(- \int_0^S \alpha(\eta_t^0) dt \right)}{\mathbf{E}'' \exp \left(- \int_0^{S_0^1} \alpha(\eta_t^1) dt \right)}
 \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}'\tilde{S}_1 &= \mathbf{E} \int_0^S \exp \left(- \int_0^t \alpha(\eta_u^0) du \right) dt \\ &+ \frac{(1 - \mathbf{E} \exp(- \int_0^S \alpha(\eta_t^0) dt)) \mathbf{E}' \int_0^{S_0^1} \exp(- \int_0^t \alpha(\eta_u^1) du) dt}{\mathbf{E}' \exp(- \int_0^{S_0^1} \alpha(\eta_t^1) dt)}. \end{aligned}$$

Therefore,

$$\lambda(\infty) = \frac{1}{D_1} \left(1 - \mathbf{E} \exp \left(- \int_0^S \alpha(\eta_t^0) dt \right) \right), \tag{4.21}$$

where

$$\begin{aligned} D_1 &= \mathbf{E} \int_0^S \exp \left(- \int_0^t \alpha(\eta_u^0) du \right) dt \mathbf{E}' \exp \left(- \int_0^{S_0^1} \alpha(\eta_t^1) dt \right) \\ &+ \left(1 - \mathbf{E} \exp \left(- \int_0^S \alpha(\eta_t^0) dt \right) \right) \mathbf{E}' \int_0^{S_0^1} \exp \left(- \int_0^t \alpha(\eta_u^1) du \right) dt. \end{aligned} \tag{4.22}$$

The expectations $\mathbf{E}' \exp(- \int_0^{S_0^1} \alpha(\eta_t^1) dt)$ and $\mathbf{E}' \int_0^{S_0^1} \exp(- \int_0^t \alpha(\eta_u^1) du) dt$ can be expressed in terms of “ordinary” expectations \mathbf{E} with the help of formulas (4.17) and (4.18) or Remark 4.3.

The estimates derived above can be used conveniently owing to the fact that, in many practical problems, one can estimate all terms involved in (4.21) and (4.22) and thus give bounds for $\lambda(\infty)$ as we will see it in the following sections.

5. Accuracy Estimates of the Asymptotic Failure Rate

In this section, we obtain accuracy estimates for the proposed approximations of the asymptotic failure rate.

5.1 Accuracy of λ^0

Let λ^0 be defined as in (3.9). Suppose that $\alpha_u = \alpha(\eta_u^0)$ and the assumptions of Conditions C0 or Remark 3.5 are satisfied. Then

$$\lambda^0 = \frac{\mathbf{E} \int_0^S \alpha(\eta_u^0) du}{\mathbf{E}S}. \tag{5.1}$$

We have already proved, in Proposition 3.3, that $\lambda^0 \geq \lambda(\infty)$. Let us estimate the relative accuracy of this approximation:

$$\rho^0 = \frac{\lambda^0 - \lambda(\infty)}{\lambda(\infty)}. \tag{5.2}$$

By (4.21), (4.22), and (5.1),

$$\rho^0 = \frac{D_1}{\mathbf{E}S} \frac{\mathbf{E} \int_0^S \alpha(\eta_u^0) du}{1 - \mathbf{E} \exp(- \int_0^S \alpha(\eta_u^0) du)} - 1. \tag{5.3}$$

Denote

$$\xi = \int_0^S \alpha(\eta_u^0) du. \tag{5.4}$$

Routine estimates show that

$$D_1 \leq \mathbf{E}S + \mathbf{E}\xi \mathbf{E}S_0^1 \tag{5.5}$$

and

$$1 - \mathbf{E} \exp \left(- \int_0^S \alpha(\eta_u^0) du \right) \geq \mathbf{E}\xi - \frac{1}{2} \mathbf{E}\xi^2. \tag{5.6}$$

Bringing together (5.1) through (5.6) and assuming that $\mathbf{E}\xi^2 < 2\mathbf{E}\xi$, we arrive at the inequality

$$\rho^0 \leq \frac{\mathbf{E}\xi \mathbf{E}S_0^1 / \mathbf{E}S + \mathbf{E}\xi^2 / 2\mathbf{E}\xi}{1 - \mathbf{E}\xi^2 / 2\mathbf{E}\xi}. \tag{5.7}$$

In practice, all relevant characteristics in (5.7) can be estimated. Namely, $\mathbf{E}S$ represents the mean inter-regeneration time for process (η_s^0) ; $\mathbf{E}S_0^1$ is the mean length of the delay of process (η_s^1) ; and ξ is the integral intensity to fail accumulated during a regeneration cycle of (η_s^0) . All these quantities can be estimated, say, by the test functions method (see Kalashnikov [15] (Chapter 50)) which uses rather general information about processes under investigation and does not require solving various equations. Corresponding examples will be given in [6]. We only note that both summands in the numerator on the right-hand side of (5.7) are small as a rule for highly reliable systems.

5.2 Accuracy of $\lambda_V(\infty)$

Let us estimate the relative accuracy

$$\rho_V = \frac{|\lambda_V(\infty) - \lambda(\infty)|}{\lambda(\infty)}. \tag{5.8}$$

For this, let us give a stationary representation for $\lambda_V(\infty)$ similar to that presented in Theorem 4.4. We know that (see (3.17))

$$\lambda_V(\infty) = \frac{1}{\pi(\mathcal{M}_b)} \int_{\mathcal{M}_b} A(\eta; \mathcal{P}) \pi(d\eta).$$

Let

$$\pi'(\cdot) = \frac{\pi(\cdot)}{\pi(\mathcal{M}_b)}$$

be the conditional stationary measure of (η_s) restricted to \mathcal{M}_b . Let us prove that π' can be regarded as the stationary measure of a regenerative process $\eta' \equiv (\eta'_s)$ and show that η' can be constructed from η^0 and its i.i.d. versions similarly to process $\tilde{\eta}$ (see (4.10) and (4.11)).

Assume that the underlying process (η_s) is *stationary*. Define the following r.v.'s (which are stopping times for (η_s)):

$$\begin{aligned} V_0 &= \min\{t: \eta_t \in \mathcal{P}, t \geq 0\}, \\ U_k &= \min\{t: \eta_t \in \mathcal{M}_b, t > V_k\}, k \geq 0, \\ V_k &= \min\{t: \eta_t \in \mathcal{P}, t > U_{k-1}\}, k \geq 1. \end{aligned}$$

Let

$$\eta(k) = \eta_{U_k}, \quad k \geq 0.$$

Evidently, $(\eta(k), U_k)_{k \geq 0}$, is a Markov renewal process.

Theorem 5.1: *Let (η_s) be an ergodic Markov process satisfying the assumptions of Proposition 3.10 and having stationary distribution $\pi(\cdot)$. Then there exists a regenerative process η' with stationary distribution π' satisfying $\pi'(\cdot) = \pi(\cdot)/\pi(\mathcal{M})$, and*

$$\lambda_V(\infty) = \int_{\mathcal{M}} \alpha(\eta)\pi'(d\eta).$$

Moreover,

$$\eta'_t = \eta_t^0 \text{ if } t < T^0, \mathbf{P}(\eta'_{T^0} \in \cdot) = p(\cdot),$$

where T^0 is a r.v. such that

$$\mathbf{P}(T^0 \leq t \mid (\eta_s^0)_{s \geq 0}) = \mathbf{P}(T^0 \leq t \mid (\eta_s^0)_{0 \leq s \leq t}) = 1 - \exp\left(-\int_0^t \alpha(\eta_u^0)du\right).$$

Proof: Since (η_s) has a stationary distribution, $\eta(k)$, $k > 1$, has a stationary distribution as well. Without loss of generality we can regard the stationary version of process (η_s) as defined on $-\infty < s < \infty$ and similarly we can define sequences U_k and V_k for negative values of k . By stationarity of (η_s) , the sequence $((\eta_s)_{U_{k-1} < s \leq U_k}, U_k)$ is a stationary marked process that is all shifts $((\eta_s)_{N(t)-1+k < s \leq U_{N(t)+k}}, U_{N(t)+k} - t)$ of time t are identically distributed for all t , where $N(t) = \min\{k: U_k > t\}$. It follows that sequence $(\eta_s)_{U_{k-1} < s \leq U_k}, k \geq 1$ is stationary and has the corresponding Palm distribution (see Franken, König, Arndt and Schmidt [12] (Theorem 1.3.1)). Hence, $\eta(k)$ is stationary. Denote its unique stationary distribution by $p(\cdot)$.

Viewing (η_s) as a semi-regenerative process with semi-regeneration times U_k , $k \geq 0$ (see Disney and Kiessler [8] and Çinlar [2] (Chapter 10, Section 6)), we can represent π in the form

$$\pi(\cdot) = \frac{\mathbf{E} \int_0^\tau I(\eta_t^p \in \cdot) dt}{\mathbf{E}\tau}. \tag{5.9}$$

In formula (5.9), (η_s^p) is a version of the Markov process (η_s) having the initial distribution $p(\cdot)$ and τ is the length of a semi-regeneration cycle of this version, that is

$$\tau = \sigma + \sigma_r,$$

where

$$\sigma = \min\{t: \eta_t^p \in \mathcal{P}, t > 0\}$$

is the first failure time for (η_s^p) and

$$\sigma_r = \min\{t: \eta_t^p \in \mathcal{M}, t > \sigma\} - \sigma$$

is the length of repair time (which is the time required for returning to the subset \mathcal{M}). It follows that

$$\pi' = \frac{\mathbf{E} \int_0^\sigma I(\eta_t^p \in \cdot) dt}{\mathbf{E}\sigma}. \tag{5.10}$$

Let us decompose process (η_s^p) into a regenerative Markov process (η_s'') and a r.v. T'' such that

$$\mathbf{P}(T'' \leq t \mid (\eta_s'')) = 1 - \exp\left(-\int_0^t A(\eta_u'', \mathcal{P})du\right). \tag{5.11}$$

This can be done due to Theorem 2.7 (see also (2.4)). Thus, (η_s'') is a version of the zero-delayed process (η_s^0) . Namely, they both have the same generator \mathbf{A}^0 (see (2.22)) but different (in general) initial distributions. It follows from (2.5) that

$$\mathbf{E} \int_0^\sigma I(\eta_t^p \in \cdot) dt = \mathbf{E} \int_0^{T''} I(\eta_t'' \in \cdot) dt. \tag{5.12}$$

Now, define a regenerative process η' using construction (4.11) but taking i.i.d. copies of η'' as η^i ($i \geq 1$). Equip process η' with regeneration times of the first type (see (4.11) where these times are shown explicitly and the construction of regeneration times given prior to Theorem 4.4). Then stationary distribution p' of process η' has the form

$$p' = \frac{\mathbf{E} \int_0^{T''} I(\eta_t'' \in \cdot) dt}{\mathbf{E} T''}. \quad (5.13)$$

By (5.10) through (5.13), we conclude that $p' = \pi'$. This completes the proof of the theorem. \square

The construction of process (η'_s) is quite similar to the construction of $(\tilde{\eta}'_s)$. Let \tilde{S}' be the first regeneration time of process η' (the analog of \tilde{S}^1 for $\tilde{\eta}$). Then, under assumptions of Theorem 5.1,

$$\lambda_V(\infty) = \frac{1}{\mathbf{E}(\tilde{S}')} \mathbf{E} \int_0^{\tilde{S}'} \alpha(\eta'_t) dt. \quad (5.14)$$

Denote by S''_0 the delay time of the version η'' (an analog of S^1_0 for η^1). By arguments like those used in deriving (4.21) and (4.22) from (4.20), we obtain from (5.14) that

$$\lambda_V(\infty) = \frac{1}{D_2} \left(1 - \mathbf{E} \exp \left(- \int_0^S \alpha(\eta_t^0) dt \right) \right), \quad (5.15)$$

where

$$\begin{aligned} D_2 = & \mathbf{E} \int_0^S \exp \left(- \int_0^t \alpha(\eta_u^0) du \right) dt \mathbf{E}'' \exp \left(- \int_0^{S''_0} \alpha(\eta_t'') dt \right) \\ & + \left(1 - \mathbf{E} \exp \left(- \int_0^S \alpha(\eta_t^0) dt \right) \right) \mathbf{E}'' \int_0^{S''_0} \exp \left(- \int_0^t \alpha(\eta_u'') du \right) dt, \end{aligned} \quad (5.16)$$

and S''_0 is the delay time of the version η'' of process η^0 constructed in the proof of Theorem 5.1 (an analog of S^1_0 for η^1).

Now, let us use formulas (4.12), (4.22), (5.15) and (5.16) to obtain an estimate of ρ_V . For this, we are in need of the following conditions.

Conditions C2a

1. Process (η_s) is an ergodic Feller process satisfying the assumptions of Theorem 2.7.
2. Conditions C0 hold true.

Conditions C2b Both (η_s) and (η_s^0) are irreducible Markov processes on a finite state space.

If (η_s) meets Condition C2a.1, then, by Theorem 2.7, $((\eta_s), \sigma)$ can be decomposed into $((\eta_s^0), (\alpha_s))$, with

$$\alpha_s = A(\eta_s^0, \mathcal{P}) \equiv \alpha(\eta_s^0).$$

Condition C2a.2 yields the possibility to use relations (4.21) and (5.15) for estimation of ρ_V .

If (η_s) is a Markov process on a finite state space, then (η_s^0) is a similar process. However, the irreducibility of (η_s) does not yield the irreducibility of (η_s^0) and vice versa. This is the reason to impose Condition C2b in order to use then Remarks 3.5 and 4.5 for estimating ρ_V without

referring to Condition C0.4.

Suppose that either Conditions C2a or C2b are satisfied. By (4.21) and (5.15),

$$\rho_V = \frac{|D_2 - D_1|}{D_2}.$$

Denote

$$\beta \equiv \frac{\beta_N}{\beta_D} = \frac{1 - \mathbf{E} \exp\left(-\int_0^S \alpha(\eta_t^0) dt\right)}{\mathbf{E} \int_0^S \exp\left(-\int_0^t \alpha(\eta_u^0) du\right) dt}, \tag{5.17}$$

$$\xi^1 = \int_0^{S_0^1} \alpha(\eta_u^1) du, \quad \xi'' = \int_0^{S_0''} \alpha(\eta_u'') du.$$

We have from (4.22) and (5.16)

$$\frac{|D_2 - D_1|}{\beta_D} \leq \max(\mathbf{E}\xi^1, \mathbf{E}\xi'') + \beta \max(\mathbf{E}S_0^1, \mathbf{E}S_0'').$$

Similarly,

$$\frac{D_2}{\beta_D} \geq \exp(-\mathbf{E}\xi'').$$

This yields

$$\rho_V \leq \exp(\mathbf{E}\xi'') (\max(\mathbf{E}\xi^1, \mathbf{E}\xi'') + \beta \max(\mathbf{E}S_0^1, \mathbf{E}S_0'')).$$

Note that in “typical” reliability problems, β_D is close to $\mathbf{E}S$ if the system is highly reliable. Using the fact that $\beta_N \leq \mathbf{E}\xi$ (where $\xi = \int_0^S \alpha(\eta_u^0) du$ is defined in (5.4)), we arrive at the bound

$$\beta \leq \frac{\mathbf{E}\xi}{\mathbf{E} \int_0^S \exp\left(-\int_0^t \alpha(\eta_u^0) du\right) dt}.$$

Using (4.21) and (4.22) (resp. (5.15) and (5.16)), we have similarly

$$\lambda(\infty) \leq \frac{1 - \mathbf{E} \exp\left(-\int_0^S \alpha(\eta_t^0) dt\right)}{\beta_D \mathbf{E} \exp\left(-\int_0^{S_0^1} \alpha(\eta_t^1) dt\right)} \leq \frac{1}{\beta_D} \exp(\mathbf{E}\xi^1) (1 - \exp(-\mathbf{E}\xi))$$

and

$$\lambda_V(\infty) \leq \frac{1}{\beta_D} \exp(\mathbf{E}\xi'') (1 - \exp(-\mathbf{E}\xi)).$$

Let us collect the estimates above as a proposition.

Proposition 5.2: *Under Conditions C2a or C2b,*

$$\frac{|\lambda_V(\infty) - \lambda(\infty)|}{\lambda(\infty)} \leq \exp(\mathbf{E}\xi'') \left(\max(\mathbf{E}\xi^1, \mathbf{E}\xi'') + \frac{\max(\mathbf{E}S_0^1, \mathbf{E}S_0'') \mathbf{E}\xi}{\mathbf{E} \int_0^S \exp\left(-\int_0^t \alpha(\eta_u^0) du\right) dt} \right), \tag{5.18}$$

where S is the first regeneration time of η^0 ,

$$\xi = \int_0^S \alpha(\eta_u^0) du, \quad \xi^1 = \int_0^{S_0^1} \alpha(\eta_u^1) du, \quad \xi'' = \int_0^{S_0''} \alpha(\eta_u'') du,$$

processes η^1 and η'' are versions of process η^0 , and S_0^1 and S_0'' are their delay times. Moreover,

$$\lambda(\infty) \leq \frac{1}{\beta_D} \exp(\mathbf{E}\xi^1)(1 - \exp(-\mathbf{E}\xi)), \tag{5.19}$$

$$\lambda_V(\infty) \leq \frac{1}{\beta_D} \exp(\mathbf{E}\xi'')(1 - \exp(-\mathbf{E}\xi)). \tag{5.20}$$

It is remarkable that estimate (5.18) is expressed in terms of mean characteristics of processes η^0, η^1 , and η'' . Because of this, it can be estimated rather easily for a wide class of processes, either analytically or by simulation.

Remark 5.3: Since S is assumed to be \mathfrak{F}_∞^0 -measurable, term β_D in (5.17) can be expressed, with the help of (2.3), as the following mean characteristic:

$$\beta_D = \mathbf{E} \int_0^S \exp\left(-\int_0^t \alpha(\eta_u^0) du\right) dt = \mathbf{E} \int_0^S \mathbf{P}(T > t \mid \mathfrak{F}_\infty^0) dt = \mathbf{E}(T \wedge S).$$

5.3 Markov process with discrete state space

The bounds derived above can be simplified if one deals with particular random processes. Let us show how they can be written for Markov processes with finite state space.

Let process (η_s) be Markov with a finite state space and A be a finite matrix of its transition rates (generator). Assume that $\eta_0 = 0 \in \mathcal{M}$ (in reliability, state 0 is often regarded as the “perfect” operating state). Then, by Theorem 2.7, process (η_s^0) is Markov evolving on \mathcal{M} with initial state 0 and matrix of transition rates

$$A^0(i, j) = A(i, j) \quad i, j \in \mathcal{M}.$$

Let us view successive entering times to state 0 as regeneration times. We know that processes (η_s^1) and (η_s'') are both Markov with the same generator A^0 but with different initial distributions. In order to keep simple notations, we denote by (η_s^0) any generic Markov process having generator A^0 and by \mathbf{E}_η the conditional expectation given $\eta_0^0 = \eta$. Let

$$\tau_0 = \inf\{t: \eta_{t-}^0 \neq \eta_t^0, \eta_t^0 = 0\}$$

be the first *returning* time of process (η_s^0) to state 0. Then

$$\max(\mathbf{E}\xi^1, \mathbf{E}\xi'') \leq \sup_{\eta \neq 0} \mathbf{E}_\eta \int_0^{\tau_0} \alpha(\eta_u^0) du \quad \text{and} \quad \max(\mathbf{E}S_0^1, \mathbf{E}S_0'') \leq \sup_{\eta \neq 0} \mathbf{E}_\eta \tau_0.$$

Since S is the first returning time of η^0 to the state 0, by similar arguments as those in Remarks 5.3 and (2.2), we arrive at the equality

$$\beta_D = \mathbf{E}_0(\sigma \wedge \sigma_0), \tag{5.21}$$

where σ_0 is the first returning time of (η_s) to the state 0.

Let us note that the right-hand side of (5.18) is also an upper bound for $|\lambda_V(\infty) - \lambda(\infty)| / \lambda_V(\infty)$. Bringing this together with the above estimates and Proposition 5.2, we arrive at the following corollary.

Corollary 5.4: *Given Condition C2b, let $\epsilon, \epsilon_0, \delta, \underline{\delta}_0$, and $\underline{\beta}_D$ be such that*

$$\sup_{\eta \neq 0} \mathbf{E} \int_0^{\tau_0} \alpha(\eta_u^0) du \leq \epsilon, \quad \mathbf{E}_0 \int_0^{\tau_0} \alpha(\eta_u^0) du \leq \epsilon_0, \quad \sup_{\eta \neq 0} \mathbf{E}_\eta \tau_0 \leq \delta, \quad \underline{\delta}_0 \leq \mathbf{E}_0 \tau_0, \quad \underline{\beta}_D \leq \mathbf{E}_0(\sigma \wedge \sigma_0).$$

Then

$$\frac{\lambda_V(\infty)}{1 + \rho_V} \leq \lambda(\infty) \leq (1 + \rho_V)\lambda_V(\infty)$$

with

$$\rho_V \leq e^{\epsilon} \left(\epsilon + \frac{\delta \epsilon_0}{\underline{\beta}_D} \right).$$

Moreover,

$$\max(\lambda(\infty), \lambda_V(\infty)) \leq \frac{1}{\underline{\beta}_D} e^{\epsilon} (1 - e^{-\epsilon_0}), \quad \lambda_0 \leq \frac{\epsilon_0}{\underline{\delta}_0}.$$

In [6], we will show how to calculate parameters ϵ , ϵ_0 , δ , $\underline{\delta}_0$ and $\underline{\beta}_D$. Sometimes it can be done explicitly, but generally they have to be estimated. These estimates may be more or less accurate. The following crude estimates are evident:

$$\epsilon = \delta \max_{\eta \neq 0} \alpha(\eta), \tag{5.22}$$

$$\epsilon_0 = \frac{\alpha(0)}{|A^0(0,0)|} + \epsilon, \tag{5.23}$$

$$\underline{\delta}_0 = \frac{1}{|A^0(0,0)|} + \sum_{\eta \neq 0} \frac{A^0(0,\eta)}{A^0(0,0)A^0(\eta,\eta)}, \tag{5.24}$$

$$\underline{\beta}_D = \frac{1}{|A(0,0)|} + \sum_{\eta \in \mathcal{A}, \eta \neq 0} \frac{A(0,\eta)}{A(0,0)A(\eta,\eta)}. \tag{5.25}$$

Value δ can be estimated with the help of test functions and the following result (see Kalashnikov [15] (Chapter 1, Corollary 1)).

Lemma 5.5: *Let V be a non-negative function such that*

$$\mathbf{A}^0 V(\eta) \leq -1, \quad \forall \eta \neq 0. \tag{5.26}$$

Then,

$$\mathbf{E}_\eta \tau_0 \leq V(\eta), \quad \forall \eta \neq 0 \tag{5.27}$$

and one can take

$$\delta = \sup_{\eta \neq 0} V(\eta). \tag{5.28}$$

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